Infinite cardinals. Recall that the term *countable* means "countably infinite" in this class.

(A1) Show that $\mathbb{Z}^3$ is countable.

(A2) Suppose a ball is tossed to the ground from a height of one meter, and assume that its speed stays constant, say 1 meter per second (remember this is a math class, not a physics class!). Suppose that each time it bounces, it rebounds to 3/4 of its previous height. If the ball bounces *ad infinitum*, what is the *total* distance that it travels? So how long does it take for the ball to bounce infinitely many times?

(B1) Dedekind defined an infinite set as *a set that has a bijection to some proper subset of itself.* For instance, the integers $\mathbb{Z}$ are in bijective correspondence with the even integers $2\mathbb{Z}$.

Prove that the Dedekind condition is indeed equivalent to having more than n elements for every $n \in \mathbb{N}$.

(B2) In class, we enumerated the power set of $\{a, b, c\}$ with columns containing the characters Y and N. Use this representation to give a diagonalization proof that $2^{\aleph_0} > \aleph_0$.

Axiomatic definitions. The definition of a vector space is a set V over a field F with addition of vectors $a : V \times V \to V$ and scalar multiplication $s : F \times V \to V$, satisfying some axioms:

- (V, a) is an additive abelian group (identity, inverses, associativity, commutativity)
- scalar multiplication distributes: $(k + \ell)(v + w) = kv + \ell v + kw + \ell w$ $\forall k, \ell \in F, v, w \in V$
- scaling commutes with multiplication in F: $k(\ell v) = (k\ell)v \ \ \forall k \in F, v \in V$

Note that the definition implicitly requires that V be closed under the operations of addition and scalar multiplication.

(A3) For each of the following sets $V_i \subset \mathbb{R}^n$,

(a) give several examples of elements of the set,

(b) write down a general form for the arbitrary element of the set, and

(c) decide whether V_i is a vector space.

If it is, choose one or two axioms that are non-trivial to verify, and show that they hold. If it is not, show which axiom is violated.

$$V_1 = \{(x, y) \in \mathbb{R}^2 : x^2 - y = 0\}$$

$$V_2 = \text{the set of solutions to the equations} \begin{cases} 2x + z = 0 \\ y + 3z = 0 \end{cases}$$

$$V_3 = \{x \in \mathbb{R} : \sin(x) = 0\}$$

(B3) Let $E = \{$continuous real functions such that $f(x) = f(-x)\}$. What are some examples of polynomials in E? How about trig functions? Is E a vector space? (Explain how addition and scalar multiplication are defined.)