The gamma function.

(A1) The modern formula for the gamma function is $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\ dt$. (Recall that an improper integral is defined as the limit as $N \to \infty$ of the proper integral up to N, when that limit exists.) First compute $\Gamma(1)$ directly. Next use integration by parts ($\int u\ dv = uv - \int v\ du$) to compute $\Gamma(2)$.

(B1) Use integration by parts to prove the *functional equation* $\Gamma(n+1) = n\cdot\Gamma(n)$. Using this and the result of (A1), prove that $\Gamma(n+1) = n!$.

(A2) This was not Euler's original formula for Γ. First, he gave an extraordinarily clever generalization of factorials. It is based on the insightful but shaky infinite product formula

$$n! = \frac{1}{1+n}\cdot\frac{2}{2+n}\cdot\frac{3}{3+n}\cdots\frac{M}{M+n}\cdots$$

This is shaky because of convergence issues. But ignoring these, explain why this gives the same answer when n is a natural number to the usual definition of $n!$.

(A3) A rigorous, convergent "interpretation" of the formula in the last problem, also found in Euler, is

$$n! = \left[\left(\frac{2}{1}\right)^n \frac{1}{1+n}\right]\cdot\left[\left(\frac{3}{2}\right)^n \frac{2}{2+n}\right]\cdot\left[\left(\frac{4}{3}\right)^n \frac{3}{3+n}\right]\cdots\left[\left(\frac{M+1}{M}\right)^n \frac{M}{M+n}\right]\cdots$$

This is convergent in the sense that the partial products limit to a value,

$$n! = \lim_{M\to\infty} \frac{M!\cdot(M+1)^n}{(n+1)(n+2)\cdots(n+M)}.$$

Use this formula to evaluate 3!.

(B2) Since Γ is defined for all real (and in fact complex) input, and $\Gamma(n+1) = (n)!$, we can use the gamma function to *define* the meaning of $z!$ for values of z other than positive integers. Use the integral formula to derive the value of $\Gamma(1/2)$, and use the functional equation and the extended definition of factorials to derive the value of $\frac{1}{2}!$.

You will want to use a substitution to put the integral into an easier form, and you may use the Gaussian integral (which we will discuss in class):

$$\int_{-\infty}^{\infty} e^{-x^2}\ dx = \sqrt{\pi}.$$

(A4) But that's not how Euler did it! He instead used the cool infinite product formula of Wallis for $\pi/4$, which you can find in the Stillwell text on p153. Use the naive product formula from (A2) and the Wallis formula to defend the result $\frac{1}{2}! = \frac{\sqrt{\pi}}{2}$. If you disagree with this answer, explain.

The partition function.

The *partition function* $p(n)$ is defined as the number of ways to express n as an unordered sum of natural numbers, so for instance $3 = 2 + 1 = 1 + 1 + 1$ gives us $p(3) = 3$.

Recall the formulas of Ramanujan giving congruences for partitions: $p(5m+4) \equiv 0 \pmod 5$, $p(7m+5) \equiv 0 \pmod 7$, $p(11m+6) \equiv 0 \pmod{11}$.

You can find a table of the first $10,000$ values of $p(n)$ online at the URL

http://www.research.att.com/~njas/sequences/b000041.txt

(Linked from the website.)

(A5) Use the methods from the Chinese Remainder Theorem chapter (§5.2) to find a number N that satisfies all three congruence relations in Ramanujan's formulas. (It should have remainder 4 when divided by 5, remainder 5 when divided by 7, and remainder 6 when divided by 11.) Now use the website to find $p(N)$ and check with a calculator that Ramanujan's formulas are correct in this case.

(B3) Before Ramanujan came along, Euler worked on $p(n)$. He found some remarkable formulas. For instance, he found the following recursive relationship:

$$p(n) = \sum_{k=1}^{n} (-1)^{k+1} \left[p\big(n - \frac{1}{2}k(3k-1)\big) + p\big(n - \frac{1}{2}k(3k+1)\big) \right].$$

To make sense of this formula, we use the convention that $p(0) = 1$ (zero is like the "empty sum" and that's the only way to write it) and $p(n) = 0$ when n is negative.

Use the formula to express $p(6)$ in terms of smaller values $p(i)$, $i < 6$. Do the same for $p(5)$ and $p(4)$. Now use the definition of partitions to count $p(n)$ for $n = 1, 2, 3, 4, 5, 6$. Verify that the formulas are correct.

(B4) Another formula of Euler's was what is called a *generating function* for $p(n)$: it is a polynomial whose coefficients are the values we're looking for. Here's how it works. First he defined a function $f(q) = \prod_{m=1}^{\infty}(1 - q^m)$. Then the generating function for the partitions is given by its inverse: $F(q) = 1/f(q)$.

(a) If you wrote out the product, you would get

$$f(q) = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + \cdots$$

Verify the first four terms of this expansion by expanding the product.

(b) Now set $F(q) = 1/f(q)$. To figure out what it looks like, write $F(q) = a_0 + a_1 q + a_2 q^2 + a_3 q^3 + \cdots$. Now since

$$F(q) \cdot f(q) = 1,$$

we can solve for the coefficients a_n. For instance, the constant term of $F(q) \cdot f(q)$ is clearly a_0, and the constant term of the right-hand side is 1, so we conclude that $a_0 = 1$. The coefficient of q in $F(q) \cdot f(q)$ is $-a_0 + a_1$, and the right-hand side has no q term, so we get $-a_0 + a_1 = 0$, and we conclude that $a_1 = 1$. Continuing in this way, find the values of a_2 through a_6. How do the a_n compare to the values $p(n)$?

And a formula for π.

(B5) Here's a final gem of Ramanujan's.

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 \cdot 396^{4k}}.$$

Remember we already had a sum formula for π via $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ but it was super-useless because it converged to the truth so extremely slowly.

On the contrary, Ramanujan's formula is amazingly fast-converging and to this day gives one of the fastest known algorithms for computing π.

Find the first and second partial sum in the formula and use a calculator to see what approximations of π they give you. Are they better or worse than the continued fraction estimate 355/113?