Stereographic projection.

(A1) Recall that *stereographic projection* from the circle to the line can be written as $f(x,y) = \frac{x}{1-y}$, and from the sphere to the plane as $F(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$. Draw the diagrams accompanying these maps.

Consider the sequences of points $a_n = n$, or $b_n = n^2$, on the real line. These sequences clearly diverge (or converge to infinity, if you like). Reverse the stereographic projection map, using the fact that $x^2+y^2=1$ on the circle, to find the value of $f^{-1}(t)$ on the circle. What is the limiting behavior of the sequences (a_n) and (b_n) on the circle?

The Fundamental Theorem of Algebra.

These exercises illustrate Gauss' proof of the FTA from §14.7. You may use a graphing calculator or computer package (if so, cite what you're using), but you can also plot these curves by hand. You need to be reasonably precise to get the desired effects.

(A2) Let $f(z) = z^2 - 1$, and write z as $x + yi$. Neatly draw an xy coordinate plane and graph $\text{Re}(f) = 0$ and $\text{Im}(f) = 0$ in different colors. Draw $x = y$ and $x = -y$ with dotted lines, and draw a large circle in a third color. Now draw a large circle centered at the origin. If the circle is large enough, the colors of the curves crossing the circle should alternate. Gauss declares that there should be at least one point of intersection between the colored curves inside the circle. Verify that this is true in this case, and that those intersection points are the zeros of the polynomial.

(B1) Do the same for $g(z) = z^3 - 1$.

Symmetry of polynomials.

(A3) Consider the polynomial $p(x) = x^4 - 14x^2 + 9 = (x^2-7)^2 - 40$. Check that its roots are $\alpha = \sqrt{2}+\sqrt{5}$, $\beta = \sqrt{2}-\sqrt{5}$, $\gamma = -\sqrt{2}+\sqrt{5}$, $\delta = -\sqrt{2}-\sqrt{5}$.

Some symmetries of the roots preserve the property of satisfying integer polynomials while others don't.

For instance, one integer polynomial relation between the roots is $\alpha\cdot\beta = -3$. If you were to just exchange β and γ, this would no longer be satisfied, so that is not a valid *symmetry of the polynomial*.

Another relation is $(\alpha+\gamma)^2 = 20$. Find two more such relations between the roots.

Draw a rectangle that is not a square, and label its vertices clockwise $\alpha, \beta, \gamma, \delta$. Explain geometrically how to get $\alpha \leftrightarrow \beta, \gamma \leftrightarrow \delta$ as a symmetry of the rectangle. Explain why $\beta \leftrightarrow \gamma$ is not a symmetry of the rectangle.

For each of the four integer relations (the two given above and the two you found), verify that the symmetries of the rectangle maintain their truth.

The distribution of the primes.

Euler gave a heuristic to show that $\sum 1/p$ diverges, which is a statement about the density of primes. (It says that there are "a lot" of primes—more than there are perfect squares, in particular, since the sum over those reciprocals is convergent.) Below is a rigorous proof by Erdös, for which you will fill in several steps.

Assume $\sum 1/p$ converges. Let the primes be labeled $p_1, p_2, \ldots$. Then there must be some cutoff value k such that $\sum_{i=k+1}^{\infty} 1/p_i < \frac{1}{2}$. We will call the primes $p_1, p_2, \ldots, p_k$ the "small primes" and the rest of them "big." Now, just by the definition of k, it follows that $\sum_{k+1}^{\infty} N/p_i < N/2$ for any positive number N whatsoever.

For any fixed value of $N \in \mathbb{N}$, let N_b be the number of positive integers $n \leq N$ that have at least one "big prime" as a divisor. Let N_s be the number of positive integers $n \leq N$ that have only small primes as their prime divisors (this count includes 1, which has no prime divisors at all).

Now we will estimate how large N_s and N_b are for various values of $N \in \mathbb{N}$.

(B2) Explain why $\lfloor N/p_i \rfloor$ counts the number of positive integers $n \leq N$ which have p_i as a divisor.

Then, using facts from above, conclude that $N_b < N/2$ for all N.

(B3) Every natural number n can be written as $n = a_n b_n^2$, letting a_n be the leftover part of the prime decomposition after taking out the biggest square (called the *square-free* part). For instance, if $n = 2^5 \cdot 3^2 \cdot 7^2$, then $a_n = 2$ and $b_n = 2^2 \cdot 3 \cdot 7$. To make sure you understand this, find the a and b values for $n = 12$ and $n = 200$. (That is, find a_{12}, b_{12}, a_{200}, and b_{200}.)

Next, for a number n which has only small primes as divisors, it follows that a_n is a product of *distinct* small primes. Explain why this shows that there are only 2^k different possible values of a_n for all possible numbers n with only small primes as divisors.

On the other hand, $b_n \leq \sqrt{n} \leq \sqrt{N}$, so there are at most $\sqrt{N}$ different values of b_n for all $n \leq N$. Thus $N_s \leq 2^k \sqrt{N}$ for any N.

(B4) Show that for a large enough value of N, we have $N_s \leq N/2$.

This means that for some value of N, both $N_s \leq N/2$ and $N_b < N/2$. Find the contradiction.