CASSINI, CASSINI!

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Cassini ovals are given parametrically by

$$\gamma(t) = (R(t)\cos t, R(t)\sin t)\,,$$

for $R(t) = \sqrt{a^2\cos(2t) + \sqrt{b^4 - a^4\sin^2(2t)}}$ and changing the relative values of a, b gives different-shaped curves in the family.

The following pages show plots of various Cassini ovals, followed by plots of their curvature $\kappa(t)$ as $0 \leq t \leq 2\pi$.

Remarks:

- For $a = 0$, you get a perfect circle.
- The time parameter t controls the angle made by the position vector and the x-axis; you can see this because $\theta = \arctan(y/x) = t$. So at the start time, $t = 0$, the position is the point on the oval which lies on the positive x-axis; the direction of motion is counterclockwise and the curve is 2π-periodic. So the polar coordinates are $r = R(t)$, $\theta = t$.
- The local maxima of κ are maxima or minima of κ_s, and hence they are *vertices* of the curves in the sense of the Four Vertex Theorem—note that this confirms that there are at least four vertices for each curve.
- Where κ has zeroes, the signed curvature is also zero (since its magnitude is the same as the curvature), and often (but not always), zeroes of κ are places where κ_s changes sign. On the curves, check to see if the osculating circle flips from one side of the curve to the other—if so, then κ_s changes sign.
- Extra-outrageously cool fact: all Cassini ovals can be obtained as planar slices of round tori (surfaces of rotation obtained by rotating circles about lines). The parameters a and b control, in some fashion, the radius of the circle and the distance of the center to the axis of rotation.

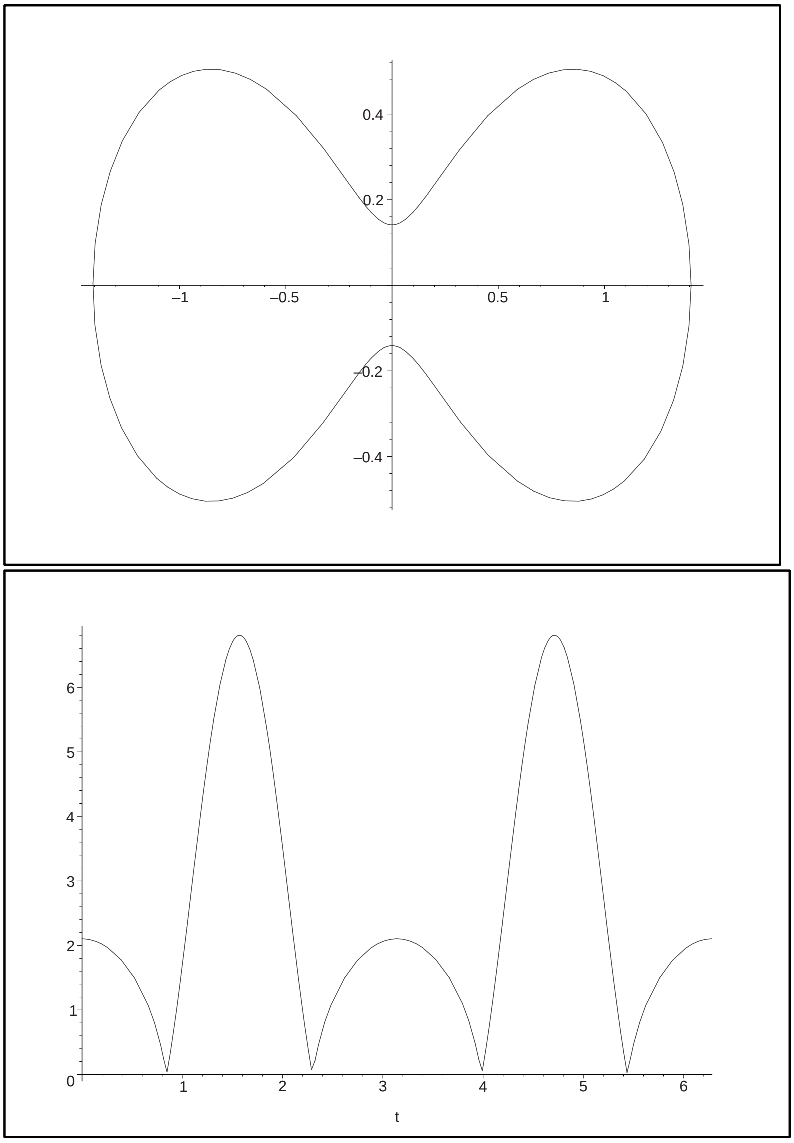

FIGURE 1. The "dumbell" ($a = .99$, $b = 1$) and its curvature.

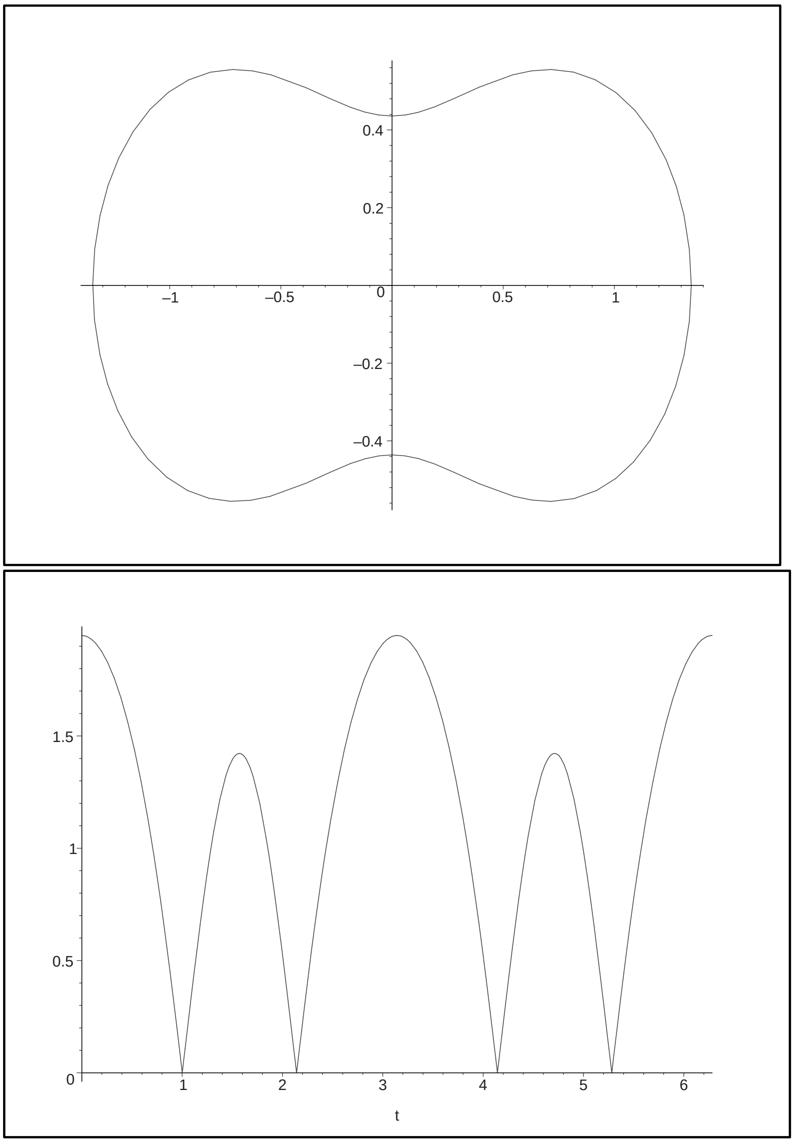

FIGURE 2. The “peanut” ($a = .99$, $b = 1$) and its curvature.

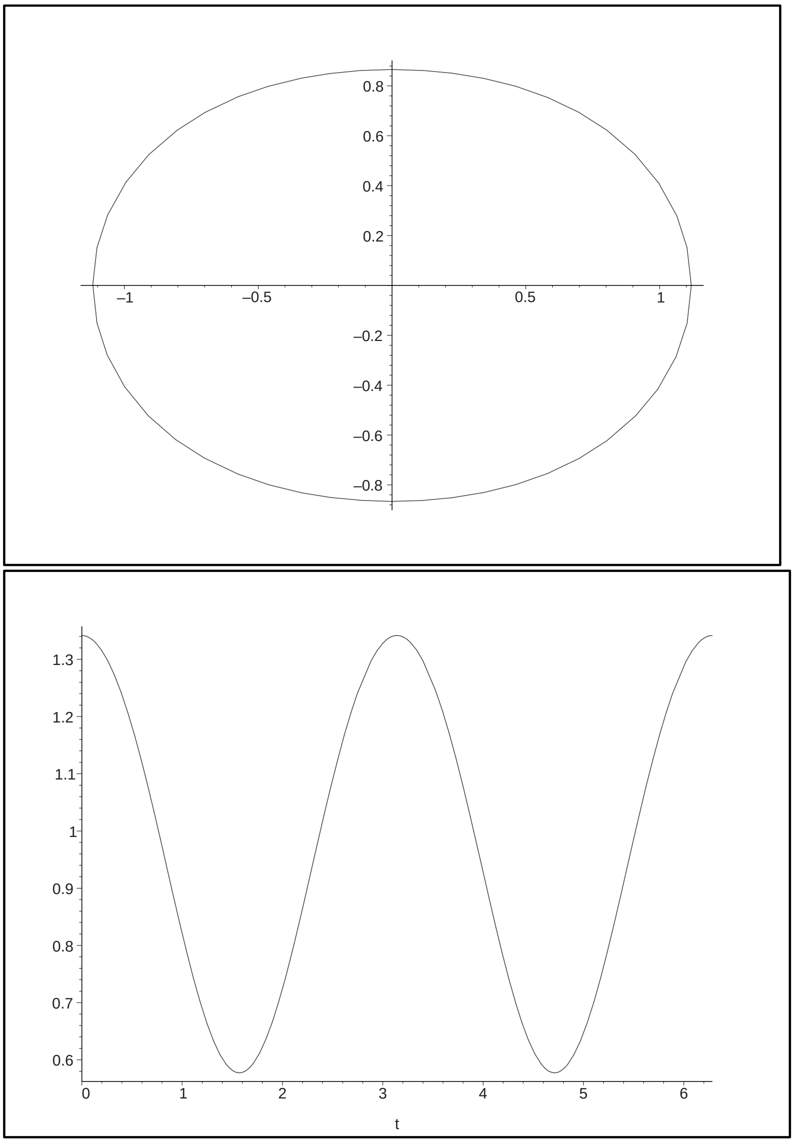

FIGURE 3. The "melon" ($a = .5$, $b = 1$) and its curvature.