ALL THE WAY WITH GAUSS-BONNET

MD

1. Some big-gun theorems

Recall that a **topological equivalence** is a bi-continuous bijection. You should think of a topological equivalence of curves/surfaces as a continuous deformation from one parametrized curve/surface to the other, never allowing crossings or "stopping" or tearing.[1]

Here are some lovely and important theorems:

Theorem 1. *The total signed curvature of a closed curve in the plane is a topological invariant:* $\int_\gamma \kappa_s$ *does not change when* γ *is replaced by a topologically equivalent curve.*

This is a version of the result sometimes known as the Hopf winding number theorem, which says that any simple closed curve in the plane has total signed curvature $\pm 2\pi$. The statements is equivalent because topological equivalence preserves simple-closed-ness. A closed curve can be thought of as a collection of simple closed curves glued together at points—topological equivalence also preserves this structure.

However, the above result would NOT be true for the (unsigned) curvature! That's because the principle of "conservation of curvature" should say that if you raise the curvature in one place, you lower it a compensating amount elsewhere. This only works if you can get positive and negative contributions to "cancel out."

Theorem 2 (Gauss-Bonnet for surfaces)**.** *The total curvature of a compact surface is a topological invariant.*

(Here, "compact" can be thought of, for surfaces in $\mathbb{R}^3$, as saying the surface is closed and bounded.)

In fact, the analogous result is true verbatim in any EVEN dimension: the total curvature of a compact n-dimensional manifold (hypersurface, if you will) is a topological invariant when n is even. (For instance, for a parametrization $\gamma : \mathbb{R}^n \to \mathbb{R}^{n+1}$, of which a parametrized surface in $\mathbb{R}^3$ is a special case.)

(In case you're curious, why just even dimensions? Because the underlying theorem—the Gauss-Bonnet theorem—says that the total curvature of a manifold only depends on the Euler characteristic, a topological invariant which is only nontrivially defined in even dimensions. A fuller answer belongs to the field of algebraic topology.)

[1] This is intentionally a bit vague since this handout is aimed at giving you an intuition for some of the structure behind results we have seen. A *homeomorphism* (bi-continuous bijection) is the most basic kind of topological equivalence, but to make the idea of continuous deformation rigorous, you need the notion of *homotopy* as well.

2. Total curvature for space curves

However, what if you are looking at total *curvature*, not total signed curvature? And what if you're interested in a space curve, not just a plane curve? Then the situation is much more complicated, and this is what homework Problems C and D were playing on.

Theorem 3 (Milnor)**.** *For any closed curve $\alpha : I \to \mathbb{R}^3$,*

$$\int_I \kappa(s)\ ds \geq 2\pi,$$

with equality if and only if the curve lies in some plane and is convex.

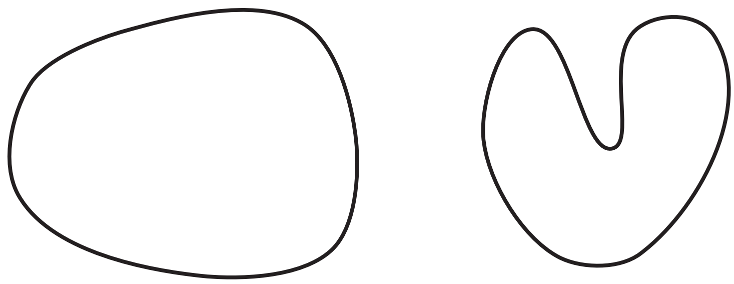

Figure 1. A plane curve is **convex** if the line segment between any two points on the curve stays in the interior. Here, the figure on the left is convex while the one on the right is not. Convexity is equivalent to the signed curvature never changing sign.

For instance, imagine a slight variation on a circle in three-space where you go around twice but stay simple (Figure 2). Of course, this forces you to have some torsion somewhere—the curve must deviate from planarity to wind around two times while staying simple.

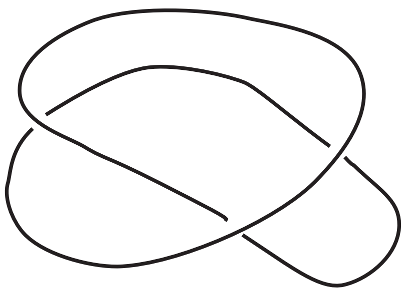

Figure 2. This simple closed curve is a continuous deformation of a circle in $\mathbb{R}^3$. However, it's got more total curvature than a planar circle: $\int \kappa(s)\ ds = 4\pi$.

Here, the total curvature would be 4π. This is, of course, topologically equivalent to a circle (which has total curvature 2π so it's clear that total curvature—and even total signed curvature—is not a topological invariant for space curves.

A **knot** is a simple closed curve in $\mathbb{R}^3$. They are considered nontrivial knots if they can't be continuously deformed to a circle (again, self-crossings are not allowed).

Theorem 4. *Every nontrivial parametrized knot has total curvature strictly greater than 4π.*

The **trefoil knot** (the one from the homework problem) is no exception. Estimating the total curvature for the parametrization

$$\alpha(t) = (4\cos 2t + 2\cos t,\ 4\sin 2t - 2\sin t,\ \sin 3t)$$

should yield ≈ 13.03524299, whereas $4\pi \approx 12.56637061$. But there is a continuous deformation of it whose total curvature is arbitrarily close to 4π ! You'll just never quite get exactly there.

On the other hand, what do you get when you project the trefoil knot into a plane? (See Figure 3.) You get a closed curve that isn't simple. Each of the two simple pieces contributes 2π to the total curvature, so the total curvature of the projection is exactly 4π. This is cool.

Remark: the title of this handout is stolen from the lovely paper [1]. It tours through the history of Gauss-Bonnet and some related results. Highly recommended.

References

[1] Daniel Gottlieb, *All the way with Gauss-Bonnet and the sociology of mathematics*, American Mathematical Monthly 103 (1996), 457–469.
URL=http://www.math.purdue.edu/~gottlieb/Papers/gauss-bonnet.pdf

[2] J.W. Milnor, *On the total curvature of knots*, Annals of Math., 2nd ser, Vol 52, No 2 (Sept. 1950), 248–257.
URL=http://www.math.ucdavis.edu/~mduchin/116/milnor.pdf

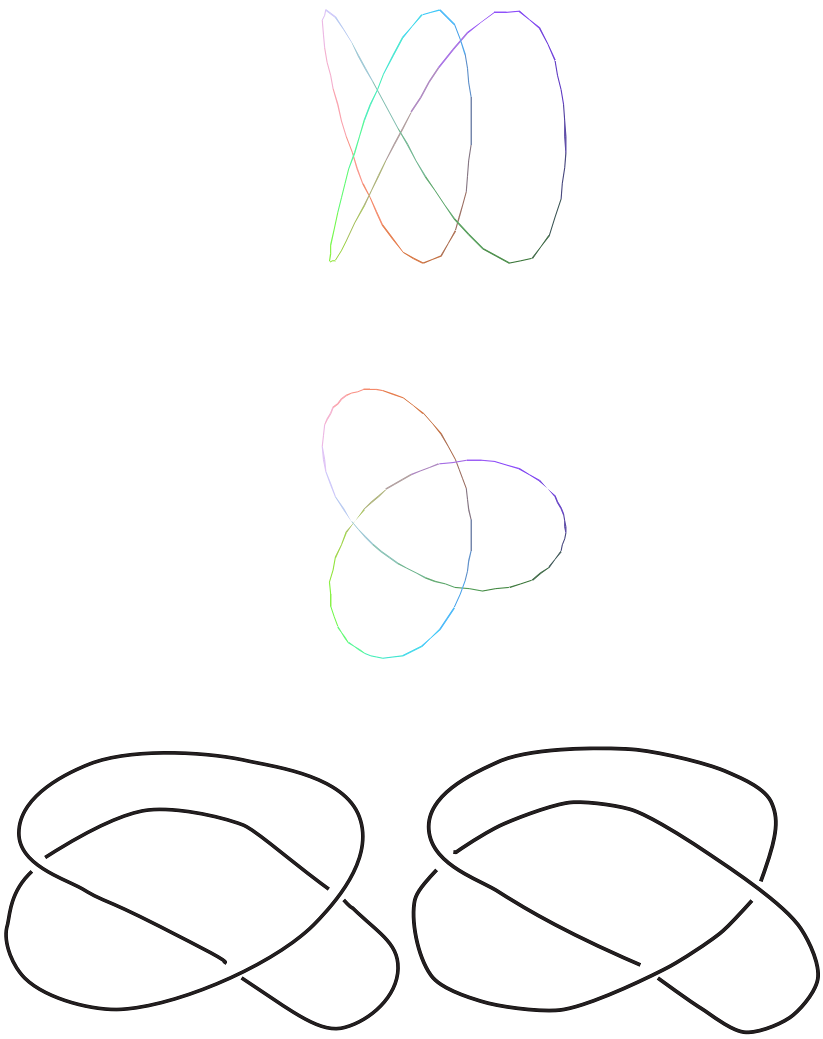

FIGURE 3. Here is a rendering of the knot; the second picture is a projection of the first picture in the "down" direction (into the xy plane), where it is no longer simple. Finally, check out the difference in the crossing patterns in the two space curves shown side-by-side. The first is topologically "unknotted" (it's the double circle mentioned above) while the second is a trefoil.