9. Introducing Fuchsian groups

9.1. Discreteness in $\text{PSL}_2(\mathbb{R})$. For the special case of subgroups of isometries of $\mathbb{H}$, discreteness is equivalent to proper discontinuity, so the study of Fuchsian groups is precisely the study of hyperbolic orbifolds via the correspondence of $\mathbb{H}$ with $\Gamma\backslash\mathbb{H}$.

**Theorem 1.** For $\Gamma \leq \text{PSL}_2\mathbb{R}$ acting on $\mathbb{H}$,

$\Gamma$ discrete $\iff$ $\Gamma$ acts properly discontinuously (orbits discrete and stabilizers finite).

Recall that in the previous section we saw that this would not be true for group actions in general, or even for $\Gamma \leq \text{PSL}_2\mathbb{R}$ on $\partial\mathbb{H}$, so this property is special to the isometric action on $\mathbb{H}$.

**Proof.** First, $\Gamma$ discrete clearly implies the stabilizers are finite. (The stabilizer of $i$ is $\Gamma \cap K$, and discrete subgroups of $K \cong S^1$ are finite. The stabilizer of any point is conjugate into $K$, so the same argument works.) Next, we will use discreteness of $\Gamma$ to deduce discreteness of orbits. Consider the orbit of $i$. Since $\Gamma$ is discrete, there is some $\epsilon$-neighborhood of the identity within which there are no other elements of $\Gamma$. If $T \in \Gamma$ is written in $NAK$ form, then $Ti = ai + b$, and $a$ and $b$ are $\epsilon$-bounded away from 1 and 0, respectively. But then $Ti$ is $\epsilon$-bounded away from $i$.

Other direction: Assume $\Gamma$ not discrete. Then there is a sequence of $T_n \in \Gamma$ such that $T_n \to I$. Then $T_n.z \to z$ for any $z$. Consider $\{T_n.z\}$. Either this set contains infinitely many distinct points in any neighborhood of $z$ or else infinitely many $T_n$ fix $z$.

Note that we’ve been using the following logic: transformations which are closer and closer to the identity move points in $\mathbb{H}$ less and less. This is certainly true, but be careful: there is NOT a uniform comparison between $||T - T'||$ and $d(Tz, T'z)$. Consider, first, the case of hyperbolic isometries. As we’ve seen, they have a pair of fixed points and an axis—the geodesic between the fixed points—which is fixed setwise as the points are moved away from the repelling and towards the attracting fixed point. Furthermore, the points on the axis move a fixed distance. If $T = \begin{pmatrix} k & 0 \\ 0 & 1/k \end{pmatrix}$, then $z \mapsto k^2z \mapsto k^4z$ and so on; for points on the axis (which in this case is the imaginary axis), they move $d(z, k^2z) = d(ai, k^2ai) = \ln(k^2) = 2\ln k$ with each application of $T$. For a point not on the axis, it is moved by more than that: if $D$ is the distance of $z$ from the imaginary axis, then $d(z, Tz)$, which is upper-bounded by $2D + 2\ln k$, is in fact an increasing function of $D$. This can be proved by either a direct computation using the distance formulae in Katok’s book, or by a more “visual” method: reduce the problem to showing that for $w = a + bi$ on the unit orthocircle ($a^2 + b^2 = 1$), $d(w, 2w)$ increases as $b$ decreases. But note that
hyperbolic circles centered at $w$ are reduced in Euclidean radius as $b$ decreases, and
argue from there. The full proof was presented in class.

Thus we can define the translation length of a hyperbolic isometry to be the
minimum distance between any point and its image under the isometry. Even
hyperbolic isometries very close to the identity move some points in $\mathbb{H}$ very far,
namely points far from the hyperbolic axis. The situation is similar for elliptics—
points far from the center of the rotation are moved far. Similarly for parabolics
and points far from the boundary fixed point. So no hyperbolic isometry has the
feature of Euclidean translation: that every point is moved by the same amount.

9.2. Mining the algebraic structure. We can now use the same basic proof
technique to deduce lots of consequences relating the algebraic structure of a Fuchsian
group to the geometry of its action: assuming a group acting on $\mathbb{H}$ is not
Fuchsian means you have a sequence of elements $T_n \to I$. So to prove that Fuchsian
groups have a certain property, you often start with such a sequence and derive
a contradiction to the property.

Definition. Recall that the centralizer of a group element $g \in G$ is the set of all
things that commute with it: $C_G(g) = \{ h \in G : gh = hg \}$.

Some Facts.
- Fixed points of elliptics do not accumulate.
  To prove this, start with $T_n z_n = z_n$ and $z_n \to z$; deduce $T_n z \to z$, a contradiction
to discreteness.
- Elements of $PSL_2 \mathbb{R}$ commute if and only if they have the same fixed point set.
  (So two hyperbolic commute if and only if they have the same axis.)
  First, say that $S$ and $T$ commute. Then let $Fix T$ be the fixed point set of $T$, and
say $Fix T = \{ t_\alpha \}$. Now, $TS(t_\alpha) = ST(t_\alpha) = S(t_\alpha)$, so $T$ fixes $S t_\alpha$, so $S t_\alpha = t_\beta$.
  Thus $S$ fixes $Fix T$ as a set, and similarly $T$ fixes $Fix S$. Case 1: either $T$ or $S$ is
elliptic. Then it has one fixed point only, so that point is fixed by the other map
also, and it must be the full fixed point set of each map, so $Fix S = Fix T$. Case 2:
either $T$ or $S$ is hyperbolic. Say $T$ is hyperbolic, and without loss of generality say
it fixes $0, \infty$. Then either $S$ fixes those points as well or $S(0) = \infty, S(\infty) = 0$. But
such a map must have the form $\left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$, whose trace is less than 2, so $S$ is elliptic,
which reverts to Case 1. Case 3: both $T$ and $S$ are parabolic. If $T$ fixes only one
boundary point, then $S$ fixes it as well, and for parabolics that is the whole fixed
point set, so $Fix S = Fix T$.

Next, we recall that $K, A, N$ are abelian; this means that if $H$ is one of the sub-
groups $K, A, N$ then for $T \in H$, we have $H \subseteq C(T)$. But on the other hand, $C(T) \subseteq
H$ because elements of $K, A, N$ are those that fix the points $\{ i \}, \{ \infty \}, \{ 0, \infty \}$, re-
spectively, and commuting elements have the same fixed points. Now if $U$ and $T$
have the same fixed point set $\mathcal{F}$, let $S$ be the element sending $\mathcal{F}$ to one of the standard
fixed point sets $\{ \{ i \}, \{ \infty \}, \{ 0, \infty \} \}$. Deduce that $U, T \in S^{-1} HS$; from that it
follows that they commute. In general, we have also shown that $C(T) = S^{-1} HS$
where $S$ “standardizes” the fixed point set of $T$ to the set characteristic of the
subgroup $H$.

- For Fuchsian groups, the following are equivalent: (1) $\Gamma$ has the same fixed
  point set for all (nonidentity) elements; (2) $\Gamma$ is abelian; (3) $\Gamma$ is cyclic.
  Abelian implies conjugate into $H$, which is homeomorphic to $S^1$ or $\mathbb{R}$, so discrete
subgroups are cyclic. The other implications should be very straightforward.
This last bullet-point is significant because it means there is no Fuchsian $\Gamma$ which is isomorphic to $\mathbb{Z}^2$—that’s a big tipoff that studying surface groups (groups of isometries of the model geometries such that the quotients are surfaces) is a good way to study the surfaces. $\mathbb{Z}^2$, you’ll recall, is the typical surface group for flat surfaces locally isometric to $\mathbb{R}^2$; the quotients are tori.

The normalizer of a subgroup $\Gamma \leq G$ is the largest subgroup of $G$ in which $\Gamma$ is normal: $N_G(\Gamma) = \{ h \in G | h\Gamma h^{-1} = \Gamma \}$. (So clearly $\Gamma$ is contained in its normalizer, and clearly the normalizer of a normal subgroup is all of $G$.) Interestingly, a way to get Fuchsian groups is to pass from a known (non-abelian) example to its normalizer.

**Claim.** If $\Gamma$ is a non-abelian Fuchsian group, then $N_G(\Gamma)$ is also a Fuchsian group.

Note that in the abelian case, this doesn’t work at all, because $N_G(\langle T \rangle)$ contains $C_G(T) = SHS^{-1}$, which is homeomorphic to either $S^1$ or $\mathbb{R}$, and in particular is not discrete. But as soon as you need at least two generators, this holds!

That’s because the proof proceeds by observing that a nondiscrete group has elements approaching the identity, and such elements in the normalizer must eventually commute with each element of $\Gamma$. But then having at least two elements in $\Gamma$ with distinct fixed point sets produces the contradiction—nothing can commute with both.

I don’t want to say very much for the moment about elementary Fuchsian groups, so I will confine the treatment here to the definition and some comments.

**Definition.** A Fuchsian group $\Gamma$ is called *elementary* if there exists a finite orbit in $\mathbb{H}$.

Recall that the *commutator* of two group elements, written $[g, h]$, is the word $ghg^{-1}h^{-1}$. If $g$ and $h$ commute, then $[g, h]$ is trivial; for this reason, the commutator can be thought of as measuring how far two elements are from commuting. The commutator is well-defined on $PSL_2\mathbb{R}$ because all four of $[\pm S, \pm T]$ are the same. This also means that not only $Tr$ but even the regular trace is defined for commutators.

Various algebraic consequences can be derived from knowing that a Fuchsian group is elementary, but the most significant is that a simple statement called Jørgensen’s inequality holds.

**Theorem 2** *(Jørgensen’s inequality).* If $\langle S, T \rangle$ is a non-elementary Fuchsian group, then

$$|Tr(T)^2 - 4| + |tr[T, S] - 2| \geq 1.$$  

One of the student presentations will discuss this inequality, including why it is useful.