MATH 180, PROBLEM SET 2

(1) What is the relationship between the continued fraction expansion of a rational number and that of its reciprocal?

(2) Choose a rational number whose continued fraction expansion has four terms: $\alpha = [a_0, a_1, a_2, a_3]$. For your choice, what are all the partial quotients, complete quotients, and convergents?

(3) As we saw in lecture, the *mediant* of a/b and c/d is defined to be $\frac{a+c}{b+d}$. The Farey graph is built by starting with an edge between two vertices labeled $1/0$ and $0/1$ and then applying the following rule: if there is an edge between a/b and c/d, then add a new edge from each of those to their mediant. (A second half of the graph has negative values, but we will not pay attention to that for this question.)

   (a) Draw part of the graph, showing at least twenty vertices.

   (b) Show that every natural number appears as a vertex in the graph.

   (c) If p/q appears as a vertex in the graph, show that q/p does as well.

   (d) For the number α you chose in the previous question, find all of its convergents as vertices in the Farey graph.

(4) The set of two-by-two matrices with integer entries and determinant one is called $SL_2(\mathbb{Z})$. Let's call two lowest-terms rational numbers a/b and c/d *special neighbors* if $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in SL_2(\mathbb{Z})$ or $\begin{pmatrix} c & a \\ d & b \end{pmatrix} \in SL_2(\mathbb{Z})$. For instance, $3/5$ is not a special neighbor of $4/5$, but $7/3$ is a special neighbor of $5/2$.

   Show that if a/b and c/d are special neighbors, then so is each of those with their mediant.