MATH 180, PROBLEM SET 6

(1) Consider the cylinder as a quotient of $\mathbb{R}^2$ by the cyclic group generated by the translation $t_{(1,0)}$. Lines in the plane can map to three different kinds of "lines" on the cylinder: horizontal lines become circles, vertical lines stay vertical lines, and other lines become spirals. We'll call these images *cylines*. (I made that up.) The following statements are all true about lines in the plane. Which of the following are true about cylines?
  (a) There is a cyline between any two points.
  (b) There is a unique cyline between any two points.
  (c) Two cylines meet in at most one point.
  (d) There are parallel cylines (pairs which do not intersect).
  (e) If you pick a cyline L and any point p that is not on it, then there is a cyline through the point p which is parallel to L.

(2) On the sphere, let a *sphline* be a great circle (that is, a circle with the same diameter as the sphere itself). Which of the statements above is true when "cyline" is replaced by "sphline"?

(3) Consider the action of the additive group $\mathbb{R}$ on itself by powers of two: $a.x = 2^a \cdot x$. Alternately, the additive group $\mathbb{Z}$ can act on $\mathbb{R}$ by the same rule. What are the orbits in each case? In each case, is the quotient space finite, infinite but countable, or uncountable?

(4) Show that the rigid motion $m = t_{-a} \circ \rho_\theta \circ t_a$ has exactly one fixed point as long as ρ_θ is nontrivial (meaning it is not the identity map).

(5) A set $S \subset \mathbb{R}^2$ is called **discrete** if there exists some $\epsilon > 0$ such that the balls of radius ϵ about all the points in S are disjoint. For instance, $\mathbb{Z}^2 \subset \mathbb{R}^2$ is discrete ($\epsilon = 1/3$ will work, say) but $\{(1, 1/n)\}$ is not because the points get arbitrarily close together. For the group $\langle t_a, t_b \rangle$ of translations, give conditions on a, b which tell you when the group has all of its orbits discrete.
Examples to help you figure this out:
  - if $a = (1,0)$ and $b = (0,1)$, then all orbits are discrete.
  - if $a = (1,0)$ and $b = (\pi, 0)$, then they are not.
  - if $a = (4,0)$ and $b = (6,0)$, then all orbits are discrete.