MATH 180, PROBLEM SET 8

Some definitions.

A **path** in a metric space (X, d) is a continuous map from part of the real line to the space. These come in three kinds: a bi-infinite path is $\gamma : \mathbb{R} \to X$; a ray is $\gamma : [a, \infty) \to X$; and a segment is $\gamma : [a, b] \to X$. For a segment γ, if the endpoints are $\gamma(a) = p$ and $\gamma(b) = q$, we will sometimes write $\gamma : p \to q$ to mean that the path connects p and q.

A map of metric spaces $f : (X, d_X) \to (Y, d_Y)$ is called an **isometry** if

$$d_X(p, q) = d_Y(f(p), f(q)) \quad \forall p, q \in X.$$

For instance, the rigid motions of the plane are exactly the isometries

$$(\mathbb{R}^2, d_{\mathrm{Euc}}) \to (\mathbb{R}^2, d_{\mathrm{Euc}}).$$

A point y is said to be **between** x and z if $d(x, y) + d(y, z) = d(x, z)$. If y is not between x and z, then you should think of it as a "detour" in traveling from x to z (it's not "on your way there").

The **length** of a path segment $\gamma : [a, b] \to X$ is the supremum over all subdivisions of the interval of the sums of the successive distances:

$$\ell(\gamma) := \sup \sum_{i=0}^{n} d\left(\gamma(x_i),\ \gamma(x_{i+1})\right),$$

where the sup is over all $a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b$. A path is **rectifiable** if this supremum is finite.

The metric space (X, d) is a **length space** if

$$d(p, q) = \inf_{\gamma : p \to q} \ell(\gamma) \quad \forall p, q \in X.$$

A path γ is a **geodesic** if it is an isometry. That is, for all real numbers a, b in the domain,

$$d(\gamma(a), \gamma(b)) = |a - b|.$$

In other words, if you think about the curve as being "traced out" in the space over a period of time, this says that for any two times you choose, the distance in space between the positions at those times is equal to the difference in times.

Turn over for the exercises.

(1) Let X_1 be the railroad metric, X_2 be the tourist metric, $X_3 = (\mathbb{R}^2, d_3)$, X_4 be the Cayley graph of the free group $F_3 = \langle a, b, c | \ \emptyset \ \rangle$, $X_5 = (C, d_{\text{Euc}})$ where C is the circle of radius 10 in $\mathbb{R}^2$, and X_6 be the torus $\mathbb{R}^2/\mathbb{Z}^2$ with its quotient metric ("pac-man" metric).

(a) Choose two of these and verify that they satisfy the axioms of a metric space.

(b) Choose a different two of these and find their group of isometries.

(c) For the final two, describe a closed metric ball of radius 2 in each. (Recall that the open balls are $B_r(p) = \{q \in X \mid d(p, q) < r\}$ and the closed balls have $d(p, q) \leq r$ instead.)

(2) Recall that the hyperbolic plane is the upper half-plane $\mathbb{H}$ together with the length metric induced by the one-form $ds = \frac{\sqrt{dx^2+dy^2}}{y}$. For the following path segments, sketch them in $\mathbb{H}$ and find their lengths. You may use an integral table if necessary, but simplify as much as possible by hand.

(a) $\alpha(t) = t - 1 + i$ on $[0, 2]$

(b) $\beta(t) = \sqrt{2}\cos t + i\sqrt{2}\sin t$ on $[\frac{\pi}{4}, \frac{3\pi}{4}]$

(c) $\gamma(t) = t + (2 - |t|)i$ on $[-1, 1]$

Of these, which is the longest? Which is the shortest?

(3) True or false: if $\gamma : [a, b] \to X$ is a geodesic, then for any time $c \in [a, b]$, the point $\gamma(c)$ is between $\gamma(a)$ and $\gamma(b)$. Explain.

(4) Consider the following metric space: $(\mathbb{R}, d_\ell)$ for $d_\ell(a, b) = \log(|b - a| + 1)$, where the logarithm is taken base 10. Verify that it is a metric space.

Now consider a path from 1 to 100. The distance between those two points is $d_\ell(1, 100) = \log 100 = 2$. Any continuous path from 1 to 100 along $\mathbb{R}$ should go through the point 35. Compute $d_\ell(1, 35)$ and $d_\ell(35, 100)$ and note that they don't add up to the total distance. Is there a midpoint in traveling from 1 to 100? (A point that is equally distant from both.) If so, is it **between** 1 and 100 in the metric space, or is it a "detour"?

Is $(\mathbb{R}, d_\ell)$ a length space?

Is there a geodesic between 1 and 100?