MATH 180, PROBLEM SET 9

Just as the length in a metric space can be given as a length element ds against which to integrate over a path, the area can be given as an area element dA. In the hyperbolic metric on $\mathbb{H}$, the area element is $dA = \frac{dx\,dy}{y^2}$, so for instance the area of the square with corners at i, $2i$, $i+1$, and $2i+1$ is

$$\int_0^1 \left(\int_1^2 \frac{dy}{y^2} \right) dx = \frac{1}{2}.$$

For two curves, the angle between them is defined to be the angle between their tangent lines. If two curves are tangent at a point, the angle between them is zero. Any two vertical lines are considered to be tangent at infinity.

(1) Suppose (X, d) is a metric space and $f : X \to Y$ is a bijection of sets. Show that f induces a **push-forward** metric on Y via

$$d_Y(p, q) := d(f^{-1}(p), f^{-1}(q)).$$

(That is, check that this satisfies the conditions of a metric.)

(2) Show that the matrix $k_\theta = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix} \right)$, acting by fractional linear transformation on $\mathbb{H}$, induces the rotation $\rho_{2\theta}$ as its action on the unit disk $\mathbb{D}$. Here are some steps for showing this.

(a) Recall that the Mobius transformation $f : \mathbb{H} \to \mathbb{D}$ is given by $f(z) = \frac{iz+1}{z+i}$. Rewrite this as a matrix F in $GL_2(\mathbb{C})$ and show that

$$Fk_\theta F^{-1} = \begin{pmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{pmatrix}.$$

(b) Using elementary facts about complex numbers, show that $(Fk_\theta F^{-1}).z = \rho_{2\theta}(z)$. (This will either require the fact that $r(\cos\theta + i\sin\theta) = re^{i\theta}$ or the use of double-angle formulas.)

(3) One version of the Gauss-Bonnet theorem gives the following formula for triangles in the hyperbolic plane: if a figure has three sides which are geodesic segments meeting in angles α, β, γ, then the area of the triangle is $\pi - \alpha - \beta - \gamma$.

(a) What is the area of the triangle bounded by the lines $x = 1/2$ and $x = -1/2$ and the unit circle? Compute it both ways (by an integral and using the angles) and verify that you get the same answer.

(b) What is the area of a triangle with all three vertices on the boundary $\partial\mathbb{H}$?

(c) Define a **rectangle** to be a figure with four geodesic segments for its sides which has four right angles. Prove that rectangles do not exist in $\mathbb{H}$.

The following problem is Extra Credit but highly recommended!

(4) Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{R})$.

(a) Find an algebraic expression for the fixed points of A on $\hat{\mathbb{C}}$.

(b) Let the trace of a fractional linear transformation be $\mathrm{Tr}(A) = |a+d|$. Show the following classification for the fixed points of A on $\mathbb{H} \cup \partial\mathbb{H}$.

$$\text{one fixed point in } \mathbb{H} \text{ and none in } \partial\mathbb{H} \iff \mathrm{Tr}(A) < 2$$
$$\text{no fixed points in } \mathbb{H} \text{ and one in } \partial\mathbb{H} \iff \mathrm{Tr}(A) = 2, A \neq \pm I$$
$$\text{no fixed points in } \mathbb{H} \text{ and two in } \partial\mathbb{H} \iff \mathrm{Tr}(A) > 2.$$

(c) Consider the matrix $M = \left(\begin{smallmatrix} 5 & 3 \\ 3 & 2 \end{smallmatrix}\right) \in SL_2(\mathbb{R})$. We'll show it's conjugate into one of the structure subgroups K, A, N. Here are the steps:

- Show that M has two fixed points in $\partial\mathbb{H}$, and they are real numbers.
- Let $\phi = (1+\sqrt{5})/2$ and $\phi' = (1-\sqrt{5})/2$. Verify that $\phi + \phi' = 1$ and $\phi - \phi' = \sqrt{5}$. Let
$$P = \begin{pmatrix} -1/\sqrt{5} & \phi'/\sqrt{5} \\ 1 & -\phi \end{pmatrix}$$
and show that $P \in SL_2(\mathbb{R})$ and P sends the fixed points of M to ∞ and 0.
- Show that PMP^{-1} fixes ∞ and 0. From this, show that this conjugate of M is of one of the three basic types (K, A, or N). Which one is it?