The geometry and ℓ^2-homology of Coxeter groups

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Q: Why spherical?

Example: Spherical Subset

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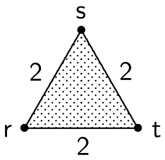

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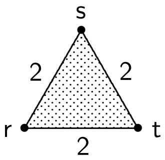

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- W_T tiles a 2-sphere with eight triangles, all angles $= \frac{\pi}{2}$.

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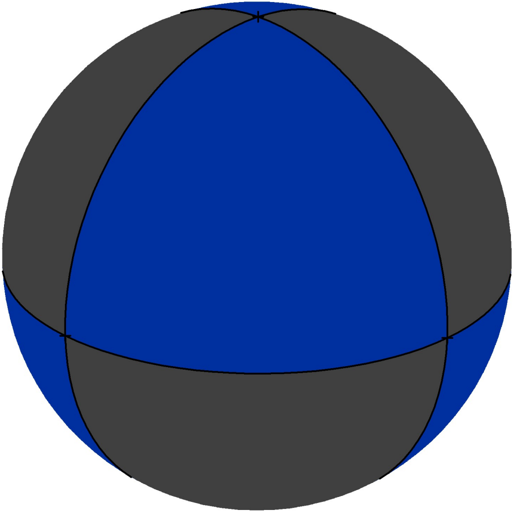

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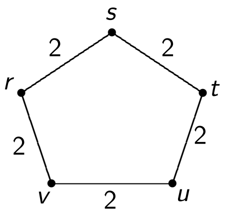
- Fact: If $L = \mathbb{S}^{n-1}$, Σ is an n−manifold.

Example: The nerve L



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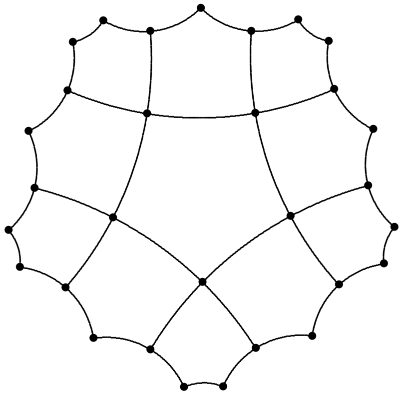

Figure: $\Sigma \cong \mathbb{H}^2$

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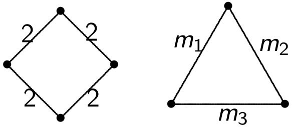
Observation

Assign dihedral angles to K so that π/m_{st} is the angle between mirrors corresponding s and t, where m_{st} is the label on the edge $\{s, t\}$.

Euclidean Examples

The 2-dimensional Euclidean groups have the following nerves:



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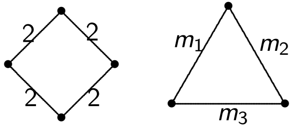

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Davis complex is the Euclidean plane tiled by squares, or $\frac{\pi}{m_1} - \frac{\pi}{m_2} - \frac{\pi}{m_3}$ triangles.
(Reflect in the faces.)

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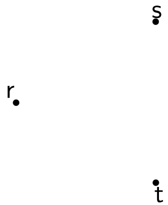

Figure: The nerve L with generators labeled

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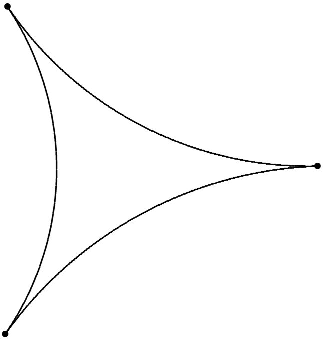

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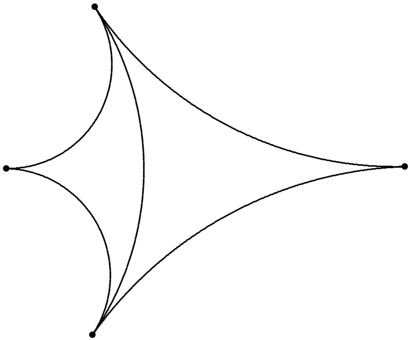

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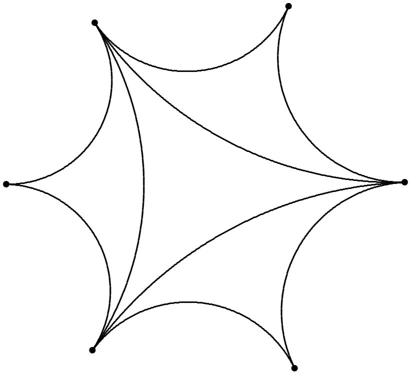

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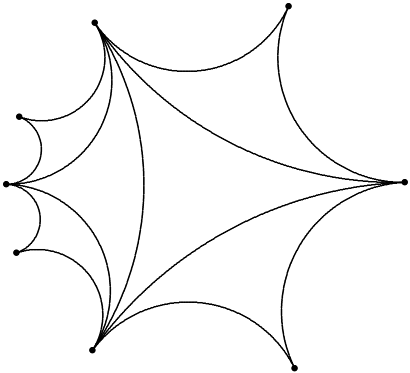

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- Assign dihedral angles as before.

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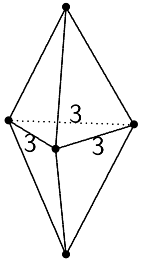

Figure: The nerve L, put 2 on non-labeled edges

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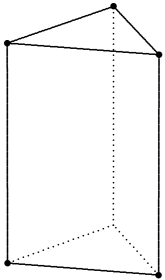

Figure: K, reflections in faces tile $\mathbb{E}^3$

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Theorem

You're handed a Coxeter system (W, S) with nerve a triangulation of $\mathbb{S}^2$. Build the appropriate K, with assigned dihedral angles. One can "cut" K along 2-dimensional Euclidean sub-orbifolds leaving pieces with the following geometries:
$\mathbb{E}^3$, $\mathbb{E}^2 \times [-1, 1]$, $\mathbb{H}^2 \times \mathbb{E}$, or $\mathbb{H}^3$.

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- Problem: It's impossible to calculate...almost.

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- Atiyah's Formula:
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- This is one of two "known" methods of calculating ℓ^2-homology.

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- Duality gives us that $\beta_1(\Sigma) = 0$

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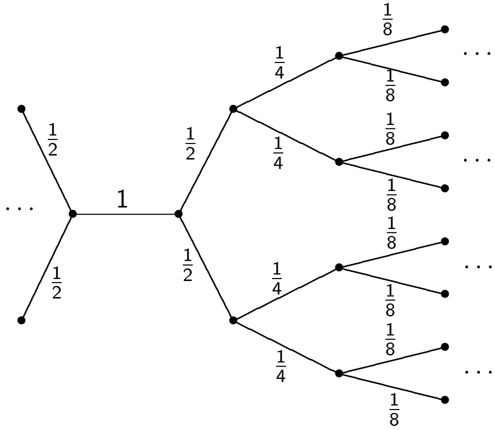

Figure: A square-summable 1-chain in Ker(∂_1)

Example: $L = P_3$, 3 points

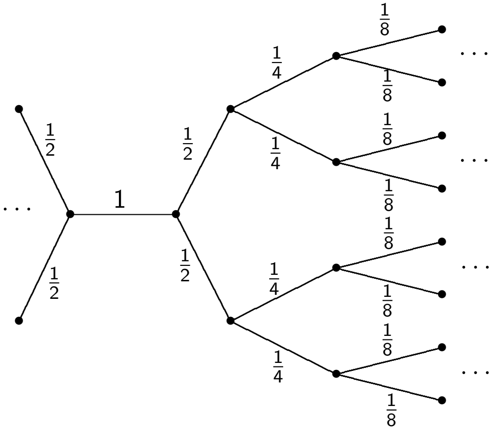

Figure: A square-summable 1-chain in $\mathrm{Ker}(\partial_1)$

Moral: $\beta_1(\Sigma_{P_2}) = 0$ while $\beta_1(\Sigma_{P_3}) \neq 0$. (In fact, $\beta_1(\Sigma_{P_3}) = \frac{1}{2}$)

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