MATH 211 — REVIEW + HIGHLIGHTS

EXTERIOR MEASURE (m_*): $\begin{cases}\text{defined as inf over (closed) cube covers.} \\ \text{realized by any almost-disjoint cube cover (if any exist)}\end{cases}$

Properties: equal to inf over open covers; monotonicity; $\aleph_0$ subadditivity; "separated additivity"

(LEBESGUE) MEASURABLE SETS

$\mathcal{B} \subseteq \mathcal{L} \subseteq \mathcal{P}$ (Borels, L.Measurable, Power set) ← these are all SIGMA-ALGEBRAS, i.e., sets of sets that are closed under $\bigcup^{\infty}$, $\bigcap^{\infty}$, ^{c}

$\mathcal{L}$ defined by open approximability (from outside): $\left(E \in \mathcal{L} \Leftrightarrow \forall \varepsilon > 0\ \exists\, \mathcal{O} \supseteq E \text{ open s.t. } m(\mathcal{O} \setminus E) < \varepsilon\right)$

Properties of $E \in \mathcal{L}$:
- closed approximability (from inside); • differs from a G_δ by negligible; • differs from an F_σ by negligible.
- $m(E) < \infty \Rightarrow$ • compact approximability (from inside);
- • finite box approximability $\left(m\left(E \,\Delta\, \bigcup_{j=1}^{N} R_j\right) < \varepsilon\right)$

MEASURABLE FUNCTION means $f^{-1}(\mathcal{B})$ is $\mathcal{L}$

EGOROV: if $\{f_n\}$ measurable on E $(m(E) < \infty)$ and $f_n \to f$ on E, then $\forall \varepsilon > 0\ \exists$ closed A_ε with $m(E \setminus A_\varepsilon) < \varepsilon$ and $f_n \to f$ uniform on A_ε

"convergence of measurable functions is uniform on a large closed set"

LUSIN: if f meas on E, $m(E) < \infty$ then $\forall \varepsilon > 0$ $\exists F_\varepsilon \subset E$ closed, $m(E \setminus F_\varepsilon) < \varepsilon$, and $f|_{F_\varepsilon}$ continuous

"measurable functions are continuous on the restriction to a large closed set"

INTEGRATION AND CONVERGENCE THEOREMS

$\int f$ is incrementally defined for simple, MBFS, $f \geq 0$, and finally all measurable.

Properties of $\int$: well-defined (indep. of choices); linear; additive; monotone; Δ inequality.

CONVERGENCE THEOREMS: for all versions, assume $f_n \to f$ a.e. and $\{f_n\}$ measurable.

$\boxed{\text{BCT}}$ if $\{f_n\}$ are MBFS with common M, E then $\int f_n \to \int f$

$\boxed{\text{Fatou}}$ if $f_n \geq 0\ \forall n$, only get $\int f \leq \liminf \int f_n$ ("sublimitivity")

$\boxed{\text{MCT}}$ if $0 \leq f_n \leq f$ then $\int f_n \to \int f$ OR if $0 \leq f_n \nearrow f$ (monotone)

$\boxed{\text{DCT}}$ if $|f_n| \leq g$, $g \in L^1$ then $\int f_n \to \int f$

$\left[\underline{\text{Cor}}: \int \sum = \sum \int \text{ for } a_n \geq 0\right]$

$L^1(\mathbb{R}^d)$ is a COMPLETE NORMED VECTOR SPACE $\left(\|f\|_1 = \int |f|\right)$

this uses $\left[f_n \to f \text{ in } L^1 \Rightarrow f_{n_k} \to f \text{ a.e.}\right]$

Useful Facts: "Internalize, don't memorize!"

Continuous on compact set $\Rightarrow$ unif. continuous; f_n unif. cont, $f_n \to f \Rightarrow f$ cont.

($\mathbb{R}$) Baire Category Theorem (countable $\cap$ of open dense sets is dense)

Borel-Cantelli Lemma (if $\sum m(E_k) < \infty$ then $m(\limsup E_k) = 0$)

Given AC, there are nonmeasurable subsets of all positive-measure sets

All Cantor sets are mutually homeomorphic ("Cantor set" means · compact · perfect · totally disconnected)

Measurable functions are a.e. limits of continuous functions

A function is Riemann-integrable iff its discontinuity set is negligible

Riemann integrable implies Lebesgue integrable, and the integrals agree.

Continuous $\Rightarrow$ measurable; Being measurable is preserved by $f_n \to f$ a.e.

Measurable functions are realizable as $\begin{cases} \text{limits of simple functions} \\ \text{OR} \\ \text{a.e. limits of step functions} \end{cases}$

$\mathbb{Q} \in F_\sigma \setminus G_\delta$

Proof Techniques

- Geometric series "trick" (for, e.g., "small" open nbhd of $\mathbb{Q}$)
- Reduce to compact case by exhaustion (e.g. $B_r \nearrow \mathbb{R}^d$) or partition (e.g. $\biguplus_n [n, n+1] = \mathbb{R}$)
- Separate integrals into large controlled part (e.g. by Egorov) and small wild part
- Uniform continuity is effectively "bounded slope"
- Can build many examples + counterexamples from homeoms. $\mathcal{C} \to \mathcal{C}'$.
- To prove a statement for measurable functions, prove for an easier class (Continuous, simple, step, MBFS) and approximate.