Math 12 Exam I Solutions Fall 2009 10/19/09

1 (12) $\int x^2 e^{2x}\,dx = \frac{1}{2}x^2e^{2x} - \int xe^{2x}dx = \frac{1}{2}x^2e^{2x} - \left[\frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}dx\right]$

$\left(\text{parts: } u=x^2 \quad dv=e^{2x};\quad du=2x\,dx \quad v=\frac{1}{2}e^{2x}\right)$

$\left(\text{parts: } u=x \quad dv=e^{2x}dx;\quad du=dx \quad v=\frac{1}{2}e^{2x}\right)$

$= \boxed{\frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{2}\left(\frac{1}{2}e^{2x}\right) + C}$

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2 (12) $\int \frac{5x^2-x-2}{(x^2+1)(x-1)}dx = \int \frac{4x+3}{x^2+1}dx + \int \frac{1\,dx}{x-1} = \int \frac{4x\,dx}{x^2+1} + \int\frac{3\,dx}{x^2+1} + \int\frac{dx}{x-1}$

$= 4\left(\frac{1}{2}\right)\ln|x^2+1| + 3\tan^{-1}x + \ln|x-1| + C$

$= 2\ln|x^2+1| + 3\tan^{-1}x + \ln|x-1| + C$

PF's: $\frac{5x^2-x-2}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}$

(*) $5x^2-x-2 = (Ax+B)(x-1) + C(x^2+1)$

$x=1$: $5-1-2 = C(2)$

$2 = 2C \Rightarrow C=1$

$5x^2-x-2 = Ax^2+Bx-Ax-B+x^2+1$

x^2: $5 = A+1 \Rightarrow A=4$

x^0: $-2 = -B+1 \Rightarrow B=3$

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3 (12) $\int \frac{1}{x}\tan^3(\ln x)\sec^3(\ln x)\,dx$

$\left(\text{Let } u=\ln x,\ du = \frac{1}{x}dx\right)$ $= \int \tan^3u\sec^3u\,du$

$= \int \underbrace{\tan^2u}_{\sec^2u-1}\sec^2u\,\sec u\tan u\,du$

$= \int(\sec^2u-1)\sec^2u\,\sec u\tan u\,du = \int(w^2-1)w^2dw$

(Let $w=\sec u$, $dw = \sec u\tan u\,du$)

$= \int(w^4-w^2)dw = \frac{w^5}{5} - \frac{w^3}{3} + C = \frac{\sec^5u}{5} - \frac{\sec^3u}{3} + C$

$= \boxed{\frac{1}{5}\sec^5(\ln x) - \frac{1}{3}\sec^3(\ln x) + C}$

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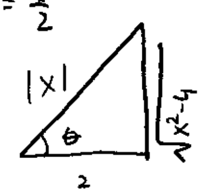
4 (12) $\int \frac{\sqrt{x^2-4}}{x^3}dx = \int\frac{2\tan\theta\, 2\sec\theta\tan\theta\,d\theta}{8\sec^3\theta}$

Let $\boxed{x = 2\sec\theta} \longrightarrow \sec\theta = \frac{x}{2}$

$dx = 2\sec\theta\tan\theta\,d\theta$

$\sqrt{x^2-4} = \sqrt{4\sec^2\theta-4} = 2\tan\theta$



$\int\frac{(2\tan\theta)(2\sec\theta\tan\theta\,d\theta)}{8\sec^3\theta}$

$= \frac{4}{8}\int\frac{\tan^2\theta\sec\theta}{\sec^3\theta}d\theta = \frac{1}{2}\int\frac{\tan^2\theta}{\sec^2\theta}d\theta$

$= \frac{1}{2}\int\frac{\sec^2\theta-1}{\sec^2\theta}d\theta = \frac{1}{2}\int\left(1-\frac{1}{\sec^2\theta}\right)d\theta$

$= \frac{1}{2}\int(1-\cos^2\theta)\,d\theta = \frac{1}{2}\int 1-\frac{1}{2}(1+\cos2\theta)d\theta$

$= \frac{1}{2}\int\left(\frac{1}{2}-\frac{1}{2}\cos2\theta\right)d\theta = \frac{1}{4}\int(1-\cos2\theta)d\theta$

$= \frac{1}{4}\left[\theta - \frac{1}{2}\sin2\theta\right] + C = \frac{1}{4}\left[\theta - \frac{1}{2}(2\sin\theta\cos\theta)\right] + C$

$= \frac{1}{4}\left[\sec^{-1}\frac{|x|}{2} - \frac{\sqrt{x^2-4}}{|x|}\cdot\frac{2}{|x|}\right] + C$

or $= \frac{1}{2}\int\frac{\sin^2\theta}{\cos^2\theta}\cdot\cos^2\theta\,d\theta$

$= \frac{1}{2}\int\sin^2\theta\,d\theta$

$= \frac{1}{2}\int\frac{1}{2}(1-\cos2\theta)\,d\theta$

$= \frac{1}{4}\int(1-\cos2\theta)\,d\theta$

$\boxed{8}$ (10) $\frac{dP}{dt} = 5P - 10, \quad P(0) = 20$

a) $\frac{dP}{dt} = 5P - 10$

$\frac{dP}{5P-10} = dt$

$\frac{1}{5}\ln|5P-10| = t + C$

$\ln|5P-10| = 5t + C$

$|5P-10| = e^{5t+c} = e^c e^{5t}$

$5P - 10 = \pm e^c e^{5t}$ Let $A = \pm e^c$

$5P - 10 = Ae^{5t}$

$5P = Ae^{5t} + 10$

$P = Ae^{5t} + 2$ (rename A)

$P(t) = Ae^{5t} + 2$

solve for A:

$P(0) = 20 = Ae^{5 \cdot 0} + 2$

$20 = A + 2 \rightarrow A = 18$

$$\boxed{P(t) = 18e^{5t} + 2}$$

b) Solve for t: $P(t) = 38$

$38 = 18e^{5t} + 2$

$36 = 18e^{5t}$

$2 = e^{5t}$

$\ln 2 = 5t$

$t = \frac{\ln 2}{5}$

$\boxed{9}$ (10)

a) $a_n = \arctan\left(\frac{n}{n+1}\right)$

$\lim_{n\to\infty} \arctan\left(\frac{n}{n+1}\right) = \arctan \lim_{n\to\infty}\left(\frac{n}{n+1}\right)$

$= \arctan \lim_{n\to\infty}\left(\frac{1}{1+\underbrace{1/n}_{0}}\right) = \arctan 1 = \pi/4$

seq. C's to $\pi/4$

b) $a_n = (-1)^n\left(\frac{n}{n+1}\right)$

(i) $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\underbrace{1/n}_{0}} = 1$

(ii) $\lim_{n\to\infty} (-1)^n\left(\frac{n}{n+1}\right)$ does not exist.

it oscillates

odd terms $\rightarrow -1$

even terms $\rightarrow 1$

$\boxed{5}$ (12) $\int_1^\infty \frac{x\,dx}{x^4+12}$ Comparison Thm: Since $0 \le \frac{x}{x^4+12} \le \frac{x}{x^4} = \frac{1}{x^3}$, $x \ge 1$

and since $\int_1^\infty \frac{dx}{x^3}$ is a convergent p-integral ($p = 3 > 1$), our integral c's. $\boxed{\text{True}}$

or: $\int_1^\infty \frac{x\,dx}{x^4+12} = \lim_{t\to\infty} \int_1^t \frac{x\,dx}{x^4+12} = \lim_{t\to\infty} \left[\frac{1}{2\sqrt{12}} \tan^{-1}\frac{x^2}{\sqrt{12}}\right]_1^t = \lim_{t\to\infty} \left[\frac{1}{2\sqrt{12}} \tan^{-1}\frac{t^2}{\sqrt{12}} - \frac{1}{2\sqrt{12}}\tan^{-1}\frac{1}{\sqrt{12}}\right]$

$\tan^{-1}\frac{t^2}{\sqrt{12}} \to \frac{\pi}{2}$

$\int \frac{x\,dx}{x^4+12} = \frac{1}{2}\int \frac{du}{u^2+12} = \frac{1}{2}\left(\frac{1}{\sqrt{12}}\right)\tan^{-1}\frac{u}{\sqrt{12}} + c$

$= \frac{1}{2\sqrt{12}} \tan^{-1}\frac{x^2}{\sqrt{12}} + c$

$u = x^2$

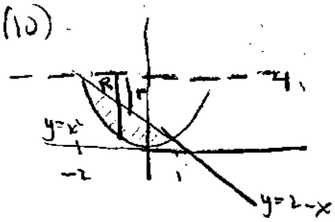
$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$= \frac{1}{2\sqrt{12}}\left(\frac{\pi}{2}\right) - \frac{1}{2\sqrt{12}}\tan^{-1}\frac{1}{\sqrt{12}}$

$\therefore$ The integral converges $\boxed{\text{True}}$

$\boxed{6}$ (10)



$V = \pi \int_{-2}^{1} (R^2 - r^2)\,dx = \pi \int_{-2}^{1} (4-x^2)^2 - (4-(2-x))^2\,dx$

$$\boxed{= \pi \int_{-2}^{1} \left((4-x^2)^2 - (2+x)^2\right) dx}$$

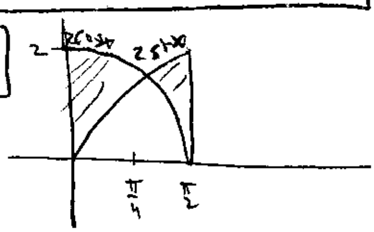
$x^2 = 2 - x$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$x = -2, x = 1$

$\boxed{7}$



$A = 2\int_0^{\pi/4} (\cos x - \sin x)\,dx + 2\int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx$

$= 2\left[(\sin x - -\cos x)\Big|_0^{\pi/4} + (-\cos x - \sin x)\Big|_{\pi/4}^{\pi/2}\right]$

$= 2\left[\left((\sin\pi/4 + \cos\pi/4) - (\sin 0 + \cos 0)\right) + \left[(-\cos\pi/2 - \sin\pi/2) - (-\cos\pi/4 - \sin\pi/4)\right]\right]$

$= 2\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1 - 1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right] = 4\sqrt{2} - 4$