1. Riemann sums: areas and averages

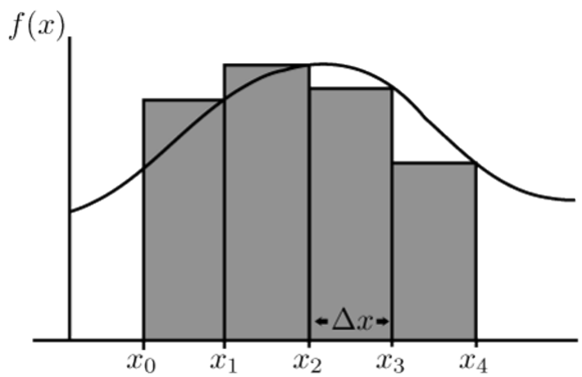
Riemann sums are introduced in your book in Section 5.1. To form a Riemann sum for a function $f(x)$ on the interval $[a, b]$ with n pieces of equal size, you would divide up the interval $[a, b]$ into n subintervals that all have the same length, $\Delta x = \frac{b-a}{n}$. (It's also possible to do a Riemann sum with subintervals of unequal length, but let's stay basic here.)

The break points of these subintervals are called *grid points* in the book, and they are as follows: first $x_0 = a$, then $x_1 = a + \Delta x$, then $x_2 = a + 2\Delta x$, and so on up to $x_n = a + n\Delta x = b$.

Then if $\overline{x}_k$ is any test point (midpoint, left endpoint, right endpoint, or any other choice) in the kth subinterval $[x_{k-1}, x_k]$, we can form the sum

$$R(n) = \sum_{k=1}^{n} f(\overline{x}_k)\cdot\Delta x = \Delta x\,[f(\overline{x}_1) + \cdots + f(\overline{x}_n)] = (b-a)\cdot\frac{f(\overline{x}_1) + \cdots + f(\overline{x}_n)}{n}.$$

Here's a picture showing an example of a right-hand Riemann sum $R(4)$:



Your book notes in Section 5.2 that $R(n)$ approximates the area under the curve $f(x)$. The reason is that $f(\overline{x}_k)\cdot\Delta x$ is the area of a rectangle with width Δx and height $f(\overline{x}_k)$, with one catch– this product can be negative, so it should be thought of as a signed area. One can see from a standard Riemann sum diagram that this sum of rectangle areas gives an estimate of the (signed) area under the curve.

We introduce the definite integral by the formula

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} R(n)$$

(for functions that are nice enough that this limit exists unambiguously), and therefore the integral can be interpreted as the area under the curve.

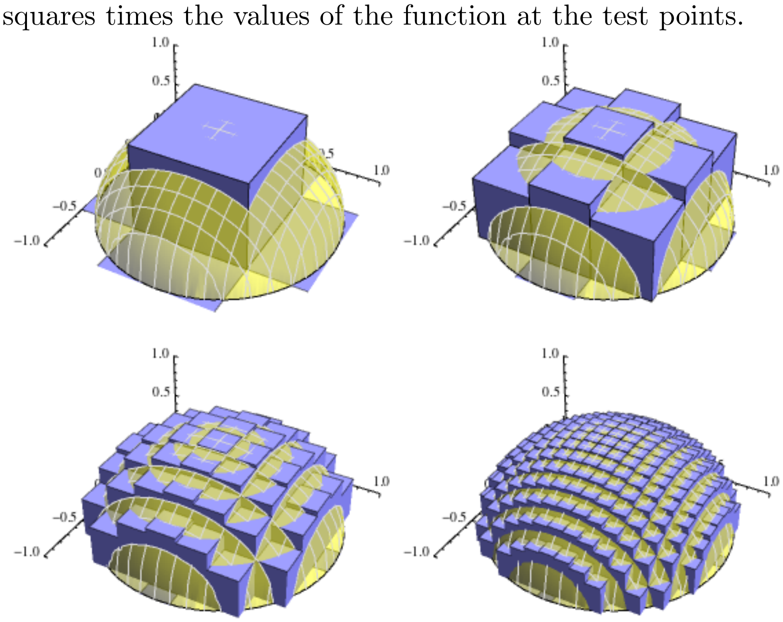
But there's also a second way to think about the meaning of the integral that is at least as important as the area interpretation. Looking back at the formula for $R(n)$, you can see that it's just $(b-a)$ times the *average* of n different values of the function $f(x)$. That suggests that $\frac{R(n)}{b-a}$ gets closer and closer to the average value of f on the interval $[a, b]$, and therefore the exact average value is equal to $\frac{1}{b-a}\int_a^b f(x)\ dx$. This turns out to be extremely useful in many applications.

2. Areas and volumes

The basic principle of a Riemann sum as an area estimate is that it chunks up a big region with a complicated boundary into small slices with simpler boundaries, as in that earlier figure. This principle can be modified for other kinds of regions, both two- and three-dimensional.

Area between two curves is treated in the book in Section 6.2. In its most basic version: if f is a curve whose graph lies above the graph of g on the interval $[a, b]$, then the difference in height at a point x is $f(x) - g(x)$, so the area between the graphs will be $\int_a^b (f(x) - g(x))\ dx$.

Now imagine that instead of finding the area under a curve, you want to find the volume under a surface. The same sort of principle is applicable: divide up the plane under the surface into a square grid, take a test point in each square of the grid, and form a Riemann sum by adding up the areas of the squares times the values of the function at the test points.



That sum will add up the volumes of the rectangular solids, which gets closer and closer to the true volume under the surface as the grid size gets small.

3. Techniques of integration: substitution

One important technique of integration is that of *substitution*, or change-of-variables.

If F is the antiderivative of f, so that $\int f(x)\ dx = F(x) + C$, then this rule can be summarized in the following formula:

$$\int f(g(x)) \cdot g'(x)\ dx = F(g(x)) + C.$$

The justification of that rule is very simple– you can check it by taking the derivative of the right-hand side, and the chain rule gives you back the integrand on the left-hand side. For definite integrals, you can use the fundamental theorem of calculus to make the rule look like this:

$$\int_a^b f(g(x)) \cdot g'(x)\ dx = F(g(b)) - F(g(a)).$$

The meaning of the rule is made more clear when you consider a change of variables $u = g(x)$. If u is defined this way, then $\frac{du}{dx} = g'(x)$, and another way to write that is $du = g'(x)dx$. This means that the rule can be rewritten as

$$\int f(u)\ du = F(u) + C,$$

which is just the fact that F is the antiderivative of f if the variable is called u instead of x! Important: if you are changing variables in a definite integral, don't forget to change the limits of integration.

Substitution becomes much clearer when you see a few examples.

- For $\int x(x^2 + 10)^{20}\ dx$, you should use $u = x^2 + 20$. Then we have $du = 2x\,dx$. So the integral becomes $\int u^{20} \cdot \frac{1}{2} du$, which is $\frac{1}{2} \cdot \frac{1}{21} u^{21} + C$, or $\frac{1}{42}(x^2 + 10)^{21} + C$.
- For $\int \cos x \sqrt{\sin x}\, dx$, you should use $u = \sin x$, so that $du = \cos x\, dx$. Then we get $\int \sqrt{u}\, du = \frac{2}{3} u^{3/2} + C = \frac{2}{3}(\sin x)^{3/2} + C$.
- Finally, consider $\displaystyle\int_0^1 \frac{e^{2x}}{e^{2x} + 1}\ dx$. Here, it's a good idea to take $u = e^{2x} + 1$, giving $du = 2e^{2x}\, dx$. So the integrand will transform to $\frac{1}{u}\frac{1}{2} du$. The original limits of integration are $x = 0$ and $x = 1$. Using the transformation rule $u = e^{2x} + 1$, we get new limits of integration $u = 2$ and $u = e^2 + 1$. So our integral becomes

$$\frac{1}{2}\int_2^{e^2+1} \frac{1}{u}\, du = \frac{1}{2}\left[\ln|u|\right]_2^{e^2+1} = \frac{\ln(e^2+1) - \ln 2}{2}.$$

See Section 5.5 for more examples.

4. Facts and Formulas

We'll expect you to have internalized certain basic facts and formulas from Pre-Calculus and Calc I, including the ones gathered here.

Logs and exponents.

$e^a e^b = e^{a+b}, \quad (e^a)^b = e^{ab}$

$\ln(xy) = \ln x + \ln y, \quad \ln \frac{x}{y} = \ln x - \ln y, \quad \ln(x^k) = k \ln x$

$(e^x)' = e^x, \quad (\ln|x|)' = \frac{1}{x}$

$(r^x)' = \left((e^{\ln r})^x\right)' = \left(e^{(\ln r)\cdot x}\right)' = \ln r \left(e^{(\ln r)\cdot x}\right) = \ln r \cdot r^x$

Trigonometry.

fundamental identities:
$\sin^2 x + \cos^2 x = 1, \quad \tan^2 x + 1 = \sec^2 x$

derivatives of all basic trig functions, e.g.,
$(\tan x)' = \sec^2 x, \quad (\sec x)' = \sec x \tan x$

derivatives of basic inverse trig, e.g.,
$(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}, \quad (\arctan x)' = \dfrac{1}{1+x^2}$

double-angle formulas:
$\cos^2 x = \dfrac{1+\cos(2x)}{2}, \quad \sin^2 x = \dfrac{1-\cos(2x)}{2}, \quad \sin(2x) = 2\sin x \cos x$