INTRODUCTION TO NILPOTENT GROUPS

Moon Duchin
YOUR NEW FAVORITE GROUP
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$s$-step nilpotent $\iff (s+1)$-fold commutators are killed
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➤ conversely: every fin-gen torsion-free nilpotent group embeds in some $UT_N(\mathbb{Z})$
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The unitriangular groups \( UT_N(Z) \) are nilpotent because addition is additive on the first nonzero superdiagonal, so taking nested commutators will terminate in at most \( N-1 \) steps

Conversely: every fin-gen torsion-free nilpotent group embeds in some \( UT_N(Z) \)

This goes through a Lie group fact: every simply connected nilpotent group is isomorphic to a Lie subgroup of some \( UT_N(\mathbb{R}) \)
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(proved by embedding the Lie algebra into strictly upper \( \Delta \)s)
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BALAYAGE: AREA VS. HEIGHT
Say a curve $\gamma=(\gamma_1,\gamma_2,\gamma_3)$ is *admissible* if its tangent vectors are horizontal, i.e., $\gamma_3' = \frac{1}{2}(\gamma_1\gamma_2' - \gamma_2\gamma_1')$. 
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- **Fact 2**: Any plane curve $\gamma=(\gamma_1, \gamma_2)$ lifts uniquely to an admissible path. Third coordinate is area.
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Proof: Stokes! \[ z = \int_{\partial R} \gamma_1 \gamma_2' - \gamma_2 \gamma_1' = \int_R dx \wedge dy = \text{Area}(R). \]
CC GEOMETRY: SUB-RIEMANNIAN AND SUB-FINSLER
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These metrics admit dilations $\delta_t(x,y,z) = (tx, ty, t^2z)$ that scale distance:

$$d(\delta_t p, \delta_t q) = t \cdot d(p, q)$$

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➤ More generally: $\gamma=(\gamma_1,\gamma_2,\gamma_3)$ as short as possible while connecting, say, $(0,0,0)$ and $(A,B,C)$. That means that $\gamma=(\gamma_1,\gamma_2)$ is shortest from $(0,0)$ to $(A,B)$ enclosing area $C$. 

\[ \text{Diagram: short path connecting points.} \]
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Let’s call these “beelines” and “area grabbers.”
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- $L^2$ case: isoperimetrix is a circle; beelines are straight horizontal lines; area-grabbers are circular spirals
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geodesic

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Note: walls are cut away to see inside—it’s a topological sphere!
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There are flat vertical “walls” coming from the beelines (range of areas for same endpoint).
Thurston’s eight 3D “model geometries”:

\[ \mathbb{R}^3, S^3, \mathbb{H}^3, S^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}, \text{Nil, Sol, and } \widetilde{\text{SL}(2,\mathbb{R})} \]
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- Complex hyperbolic space \( \mathbb{C}H^2 \): horospheres have Nil geometry.
- Higher-dimensional \( \mathbb{C}H^n \): horospheres are higher Heisenberg groups.
LATTICES IN THE LARGE: PANSU’S THEOREM
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SHAPES OF BALLS AND SPHERES

➤ Pansu is telling you that CC spheres are a close approximation to spheres in the word metric.

➤ Should still wonder: are CC geodesics good approximations of geodesics in the word metric?

➤ The CC group is divided into two parts (the beelines/walls and the area-grabbers/roof). What about the discrete group?
GROMOV'S ASK: CC SPACES “FROM WITHIN”
What are the qualitative features of a CC metric? How would you know you are in one?
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How about polygonal CC metrics on $H$?

- Visually, from a basepoint, space is divided up into two regimes (walls and roof), with rational proportion: many “$p/q$ laws.”
GROWTH SERIES AND RATIONALITY

Moon Duchin
Motivating example: The group $\mathbb{Z}^2$ has standard generators $(\pm 1, 0)$, $(0, \pm 1)$. There are also non-standard generators, like the chess-knight moves $(\pm 2, \pm 1), (\pm 1, \pm 2)$.
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We can write $\beta_n=\#B_n$, $\sigma_n=\#S_n$ for the point count of balls and spheres in the word metric. As functions of $n$, these are called growth functions of $(G,S)$ for a group $G$ and generating set $S$. 
GROWTH FUNCTIONS IN GROUP THEORY

Growth functions depend on generators \((G,S)\), but change of genset preserves growth rate, so polynomiality (and degree) is an invariant of \(G\), and so is exponential growth.
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**Theorem** (Stoll 1996): *$H_5$ has rational growth in one generating set but transcendental in another!*
### SUMMARY OF RATIONALITY RESULTS

<table>
<thead>
<tr>
<th>For all S</th>
<th>For at least one S</th>
<th>For no S</th>
</tr>
</thead>
<tbody>
<tr>
<td>hyperbolic groups</td>
<td>some automatic groups</td>
<td>unsolvable word problem</td>
</tr>
<tr>
<td>virtually abelian groups</td>
<td>Coxeter groups, standard S</td>
<td>intermediate growth</td>
</tr>
<tr>
<td><strong>Heisenberg group</strong> $H$</td>
<td>$H$, standard $S$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$H_5$, cubical $S$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$BS(1,n)$, standard $S$</td>
<td></td>
</tr>
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- Many more in middle category: some more $BS(p,q)$ examples plus “higher BS groups,” quotients of triangular buildings, some amalgams and wreath products, some solvable groups, relatively hyperbolic groups, ...
“We shall... show that the global combinatorial structure of such groups is particularly simple in the sense that their Cayley group graphs (Dehn Gruppenbilder) have descriptions by linear recursion. We view this latter result as a promising generalization of small cancellation theory... The result also indicates that cocompact, discrete hyperbolic groups can be understood globally in the same sense that the integers \( \mathbb{Z} \) can be understood: feeling, as we do, that we understand the simple linear recursion \( n \to n+1 \) in \( \mathbb{Z} \), we extend our local picture of \( \mathbb{Z} \) recursively in our mind’s eye toward infinity. One obtains a global picture of the arbitrary cocompact, discrete hyperbolic group \( G \) in the same way: first, one discovers the local picture of \( G \), then the recursive structure of \( G \) by means of which copies of the local structure are integrated.”
Let’s go back and see why lattice counts and Ehrhart polynomials were related to growth functions for $\mathbb{Z}^2$. You can get a coarse estimate of $\beta_n$ by figuring out the shape of the cloud of points $B_n$ and counting lattice points inside it.
WORD METRICS HAVE “LIMIT SHAPES” YOU CAN COUNT WITH

To see that you get an accurate first-order estimate from the Ehrhart polynomial, it suffices to show that almost all lattice points in the $n^{th}$ dilate are reached in $n$ steps.

This works well here; in general, the large spheres very closely resemble an annular shell at the boundary of your defining polygon.
<table>
<thead>
<tr>
<th>((G, S))</th>
<th>(\beta_n) ((n \gg 1))</th>
<th>(\Omega)</th>
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<tr>
<td>((\mathbb{Z}, \text{std}))</td>
<td>2(n + 1)</td>
<td>—</td>
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<tr>
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<tr>
<td>((\mathbb{Z}^2, \text{hex}))</td>
<td>3(n^2 + 3n + 1)</td>
<td>(\Box)</td>
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<tr>
<td>((\mathbb{Z}^2, \text{chess}))</td>
<td>14(n^2 - 6n + 5)</td>
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<tr>
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<tr>
<td>((F_2, \text{std}))</td>
<td>2( \cdot 3^n - 1)</td>
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GROUP GROWTH MEETS LATTICE COUNTING

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Caveat: only accurate to first order!
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$$\sigma_n = (31n^3 - 57n^2 + 105n + c_n)/18,$$

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(not just bounded above and below like $An^3 \leq \sigma_n \leq Bn^3$, which is classical)
GAME PLAN FOR HEISENBERG PANRATIONALITY

➤ We produce a finite collection of languages that we call *shapes* and *patterns* that surject onto $H$.

➤ We show that there are “rational competitions” that determine a single shape or pattern as the “winner” for each group element.

➤ We show that enumerating the winning spellings by length is a rational function for each shape and pattern.

➤ Conclusion: overall growth function is a sum of finitely many rational functions, so it is rational.
MOTIVATING EXAMPLE: KNIGHTS ON AN INFINITE CHESSBOARD
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For that sector, 3 patterns suffice:

- $a_1^* a_2^*$
- $a_3 a_8 a_1^* a_2^*$
- $a_3^2 a_8^2 a_1^* a_2^*$
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CHESS KNIGHTS AND RATIONAL COMPETITION

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Here, $G_w(n) = \{(n-1,2n-3), \ldots, (2n-3,n-1)\}$ for $n \geq 2$. 
MAJOR TOOL: COUNTING IN POLYHEDRA

Let an elementary family \( \{E(n)\} \) in \( \mathbb{Z}^d \) be defined by finitely many equalities, inequalities, and congruences as below, where the \( b \) are affine in \( n \).

A **bounded polyhedral family** \( \{P(n)\} \) is a finite union of finite intersections of these in which each \( P(n) \) is bounded.

**Theorem** (Benson): if \( f: \mathbb{Z}^d \rightarrow \mathbb{Z} \) is polynomial and \( \{P(n)\} \) is a bounded polyhedral family, then

\[
F(x) = \sum_{n=0}^{\infty} \sum_{v \in P(n)} f(v) x^n
\]

is a rational function.
APPLICATION: HEISENBERG LATTICE COUNTING

**Theorem** (D-Mooney 2014): *For any Heisenberg generators, the number of lattice points in CC balls is quasipolynomial.*
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\[
\mathbb{S}(x) = \sum_n \sigma_n x^n = \sum_n \sum_w \sum_{G_w(n)} x^n
\]
LEMMAS FOR HEISENBERG CASE

➤ Balancing Lemma: Busemann’s polygon is isoperimetrically optimal, and area falls off quadratically when you unbalance the sidelengths.
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➤ **Balancing Lemma**: Busemann’s polygon is isoperimetricaly optimal, and area falls off quadratically when you unbalance the sidelengths.

➤ **Shape Lemma**: every group element in the “roof” can be geodesically represented by something that fellow-travels an area-grabber CC geodesic.

➤ **Pattern Lemma**: every group element in the “walls” can be geodesically represented by something that fellow-travels a beeline CC geodesic.

➤ **Competition Lemma**: a winning shape or pattern for each \((a,b)\) position is determined by finitely many linear equalities, inequalities, and congruences.
SUBTLETIES
There exist word geodesics that don’t fellow-travel any CC geodesic! But every group element is represented by some word geodesic that does. We prove this algorithmically, by starting with an arbitrary word geodesic and “balancing” it at the same total length.
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This is a big problem for writing down which group elements are represented by a shape: the rational competition doesn’t allow you to check a quadratic equation. Cute idea to get around this: when two shapes compete, the *difference* in their heights is a linear polynomial, so you ascertain that they reach the same height by checking \(linear=0\).
ASSEMBLING THE INGREDIENTS

\[ S_{\text{reg}}(x) = \sum_{\omega} \sum_{n=0}^{\infty} \sum_{\Delta=0}^{K} \sum_{G_{\omega}^\Delta(n)} x^\Delta x^n, \]
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...et voilà.
GEOMETRY OF WORDS
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» TRUE for Heisenberg: first prove for polygonal CC metrics, then use bounded difference of word and CC.
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Theorem (D-Lelièvre-Mooney): If $X$ is a non-elementary hyperbolic group with any genset, then $E = 2$.

We get: all nilpotent groups have $E < 2$. Proof: CC sphere carries a limit measure that is absolutely continuous with Lebesgue, so there are positive-measure patches with $d(x,y)$ bounded away from $2n$. 
ASYMPTOTIC DENSITY

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**Figure 6.** The distortion profiles for $K = \{(*,0,*)\}$ in $H(\mathbb{Z})$ with two different generating sets. The value $d = 1$ is plotted as white, going red as $d \to \infty$. 
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- (But 25 pages of combinatorics can be replaced with a quick hit of CC geometry.)
EQUATION-SOLVING IN GROUPS
The equation problem: is there an algorithm that can decide whether solutions exist in a group to an equation given in constants and variables? A system of equations? A system of equations and inequations?
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Proof: use Mal’cev coordinates to reduce solvability to solving a quadratic equation over a lattice. OTOH, show that systems can encode arbitrary polynomials and quote Hilbert’s 10th problem!
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