

proof. They absorbed the idea that one could not consider a mathematical problem solved unless one could demonstrate that the solution was valid. How does one demonstrate this, particularly for an algebra problem? The answer seemed clear. The only real proofs were geometric. After all, it was geometry that was found in Greek texts, not algebra. Hence Islamic scholars generally set themselves the tasks of justifying algebraic rules, either the ancient Babylonian ones or new ones that they themselves discovered, and justifying them through geometry.

7.2.1 The Algebra of al-Khwārizmī and ibn Turk

One of the earliest Islamic algebra texts, written about 825 by al-Khwārizmī, was entitled *Al-kitāb al-muḥtaṣar fī ḥisāb al-jabr wa-l-muqābala* (*The Condensed Book on the Calculation of al-Jabr and al-Muqābala*) and ultimately had even more influence than his arithmetical work. The term *al-jabr* can be translated as “restoring” and refers to the operation of “transposing” a subtracted quantity on one side of an equation to the other side where it becomes an added quantity. The word *al-muqābala* can be translated as “comparing” and refers to the reduction of a positive term by subtracting equal amounts from both sides of the equation. Thus, the conversion of $3x + 2 = 4 - 2x$ to $5x + 2 = 4$ is an example of *al-jabr*, and the conversion of the latter to $5x = 2$ is an example of *al-muqābala*. Our own word “algebra” is a corrupted form of the Arabic *al-jabr*. It came into use when this and other similar treatises were translated into Latin. No translation was made of the word *al-jabr*, which thus came to be taken for the name of this science.

Al-Khwārizmī explained in his introduction why he came to write his text:

That fondness for science, by which God has distinguished the Imam al-Ma'mūn, the Commander of the Faithful, . . . that affability and condescension which he shows to the learned, that promptitude with which he protects and supports them in the elucidation of obscurities and in the removal of difficulties, has encouraged me to compose a short work on calculating by *al-jabr* and *al-muqābala*, confining it to what is easiest and most useful in arithmetic, such as men constantly require in cases of inheritance, legacies, partition, law-suits, and trade, and in all their dealings with one another, or where the measuring of lands, the digging of canals, geometrical computation, and other objects of various sorts and kinds are concerned.⁷

Al-Khwārizmī was interested in writing a practical manual, not a theoretical one. Nevertheless, he had already been sufficiently influenced by the introduction of Greek mathematics into the House of Wisdom that even in such a manual he felt constrained to give geometric proofs of his algebraic procedures. The geometric proofs, however, are not Greek proofs. They appear to be, in fact, very similar to the Babylonian geometric arguments out of which the algebraic algorithms grew. Again, like his oriental predecessors, al-Khwārizmī gave numerous examples and problems, but the Greek influence showed through in his systematic classification of the problems he intended to solve, as well as in the very detailed explanations of his methods.

Al-Khwārizmī began by noting that “what people generally want in calculating . . . is a number,”⁸ the solution of an equation. Thus the text was to be a manual for solving equations. The quantities he dealt with were generally of three kinds, the square (of the unknown), the root of the square (the unknown itself), and the absolute numbers (the



FIGURE 7.3
Al-Khwārizmī on a stamp
from the former Soviet
Union.

Biography

MUḤAMMAD IBN MŪSĀ AL-KHWĀRIZMĪ (C. 780–850)

Al-Khwārizmī, (Fig. 7.3), or perhaps his ancestors, came from Khwarizm, the region south of the Aral Sea now part of Uzbekistan and Turkmenistan. Al-Khwārizmī was one of the first scholars in the House of Wisdom founded by the caliph al-Ma'mūn, and also was one of the astronomers called to cast a horoscope for the dying caliph al-Wāthiq in 847. The story is told that although al-Khwārizmī assured the caliph he would live another fifty years, in fact the caliph died ten days later. Perhaps al-Khwārizmī felt it was not good policy to be the bearer of bad news to one's ruler. Besides the contributions to mathematics detailed in the text, al-Khwārizmī wrote a work on geography in which he developed a map of the Islamic world much superior to that known from the work of Ptolemy.

constants in the equation). He then noted that there are six types of equations that can be written using these three kinds of quantities:

1. Squares equal to roots ($ax^2 = bx$)
2. Squares equal to numbers ($ax^2 = c$)
3. Roots equal to numbers ($bx = c$)
4. Squares and roots equal to numbers ($ax^2 + bx = c$)
5. Squares and numbers equal to roots ($ax^2 + c = bx$)
6. Roots and numbers equal to squares ($bx + c = ax^2$)

The reason for the six-fold classification is that Islamic mathematicians, unlike the Hindus, did not deal with negative numbers. In their system, coefficients, as well as the roots of the equations, had to be positive. The types listed are the only types that have positive solutions. Our standard form $ax^2 + bx + c = 0$ would make no sense for Al-Khwārizmī, because if the coefficients were all positive, the roots could not be.

Al-Khwārizmī's solutions to the first three types of equations are straightforward. We only need note that 0 is not considered as a solution to the first type. His rules for the compound types of equations are more interesting. We present his solution to type 4. Because al-Khwārizmī used no symbols, we will follow him in writing everything out in words, including the numbers of his example: "What must be the square which, when increased by ten of its own roots, amounts to thirty-nine? The solution is this: You halve the number of roots, which in the present instance yields five. This you multiply by itself; the product is twenty-five. Add this to thirty-nine; the sum is sixty-four. Now take the root of this which is eight, and subtract from it half the number of the roots, which is five; the remainder is three. This is the root of the square which you sought for."⁹

Al-Khwārizmī's verbal description of his procedure is essentially the same as that of the Babylonian scribes. Namely, in modern notation, the solution of $x^2 + bx = c$ is

$$x = \sqrt{\left(\frac{b}{2}\right)^2 + c} - \frac{b}{2}.$$

Al-Khwārizmī's geometric justification of this procedure also demonstrates his Babylonian heritage. Beginning with a square representing x^2 , he adds two rectangles, each of width five ("half the number of roots") (Fig. 7.4). The sum of the area of the square and the two rectangles is then $x^2 + 10x = 39$. One now completes the square with a single square of area 25 to make the total area 64. The solution $x = 3$ is then easily found. This geometric description corresponds to the Babylonian description of the solution of $x^2 + \frac{4}{3}x = \frac{11}{12}$. (See Chapter 1, p. 38 and Fig. 1.29)

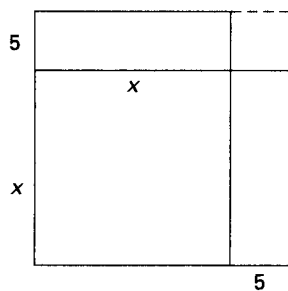


FIGURE 7.4
Al-Khwārizmī's
geometric justification for
the solution of
 $x^2 + 10x = 39$.

Although al-Khwārizmī's geometric description of his method appears to have been taken over from Babylonian sources, he or his (unknown) predecessors in this field succeeded in changing the focus of quadratic equation solving away from actually finding sides of squares and into finding numbers satisfying certain conditions. For example, he explains the term "root" not as a side of a square but as "anything composed of units which can be multiplied by itself, or any number greater than unity multiplied by itself, or that which is found to be diminished below unity when multiplied by itself."¹⁰ Also, his procedure for solving quadratic equations of type 4, when the coefficient of the square term is other than one, is the arithmetical method of first multiplying or dividing appropriately to make the initial coefficient one, and then proceeding as before. Al-Khwārizmī even admits somewhat later in his text, when he is discussing the addition of the "polynomials" $100 + x^2 - 20x$ and $50 + 10x - 2x^2$, that "this does not admit of any figure, because there are three different species, i.e., squares and roots and numbers, and nothing corresponding to them by which they might be represented. . . [Nevertheless], the elucidation by words is easy."¹¹

Finally, al-Khwārizmī's presentation of the method and geometric description for type 5, squares and numbers equal to roots, shows that, unlike the Babylonians, he could deal with an equation with two positive roots, at least numerically. In this case, $x^2 + c = bx$, his verbal description of the solution procedure easily translates into our formula

$$x = \frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - c}. \quad (7.1)$$

In fact, he states that one can employ either addition or subtraction to get a root and also notes the condition on the solution: "If the product [of half the number of roots with itself] is less than the number connected with the square, then the instance is impossible; but if the product is equal to the number itself, then the root of the square is equal to half of the number of roots alone, without either addition or subtraction."¹² The geometric demonstration in this case, which reminds us of the Babylonian description for the system $x + y = b, xy = c$ into which they would have converted this equation (see Chapter 1, p. 36 and Fig. 1.27), only deals with the subtraction in Eq. (7.1). In Fig. 7.5, square $ABCD$ represents x^2 , and rectangle $ABNH$ represents c . Therefore HC represents b . Bisect HC at G , extend TG to K so that $GK = GA$, and complete the rectangle $GKMH$. Finally, choose L on KM so that $KL = GK$ and complete the square $KLRG$. It is then clear that rectangle $MLRH$ equals rectangle $GATB$. Since the area of square $KMNT$ is $(\frac{b}{2})^2$ and that square less square $KLRG$ equals rectangle $ABNH$ or c , it follows that square $KLRG$ equals $(\frac{b}{2})^2 - c$. Since the side of that square is equal to AG , it follows that $x = AC = CG - AG$ is given by Eq. (7.1) using the minus sign. Although al-Khwārizmī briefly noted that CR could also represent a solution, he did not demonstrate this by a diagram, nor did he deal in his diagram with the special conditions mentioned in his verbal description.

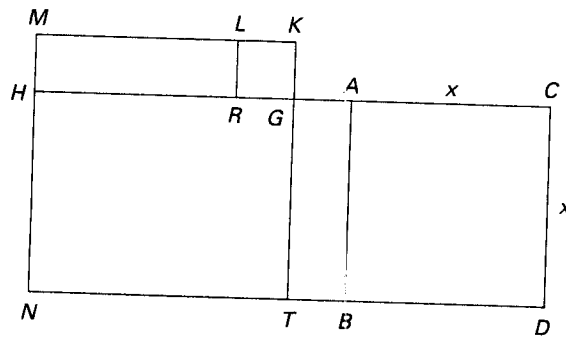


FIGURE 7.5
Al-Khwārizmī's
geometric justification
for the solution of
 $x^2 + c = bx$.

Al-Khwārizmī's text contains the word "condensed" in the title, which suggests that there existed other books that gave a more detailed discussion of algebraic procedures and their attendant geometric justifications. There is, however, only a fragment of such a work now extant, the section *Logical Necessities in Mixed Equations* from a longer work *Kitāb al-jabr wa'l muqābala* by 'Abd al-Ḥamīd ibn Wāsi ibn Turk al-Jīlī, a contemporary of al-Khwārizmī about whom very little is known. The sources even differ as to whether ibn Turk was from Iran, Afghanistan, or Syria.

In any case, the extant chapter of ibn Turk's book deals with quadratic equations of al-Khwārizmī's types 1, 4, 5, and 6 and includes a much more detailed geometric description of the method of solution than is found in al-Khwārizmī's work. In particular, in the case of type 5, ibn Turk gave geometric versions for all possible cases. His first example is the same as al-Khwārizmī's, namely $x^2 + 21 = 10x$, but he began the geometrical demonstration by noting that G , the midpoint of CH , may be either on the line segment AH , as in al-Khwārizmī's diagram, or on the line segment CA of Fig. 7.6. In this case, squares and rectangles are completed, similar in form to those in Fig. 7.5, but the solution $x = AC$ is now given as $CG + GA$, thus using the plus sign in Eq. (7.1). In addition, ibn Turk

discussed what he called the “intermediate case,” where the root of the square is exactly equal to half the number of roots. His example for this situation is $x^2 + 25 = 10x$; the geometric diagram then simply consists of a rectangle divided into two equal squares.

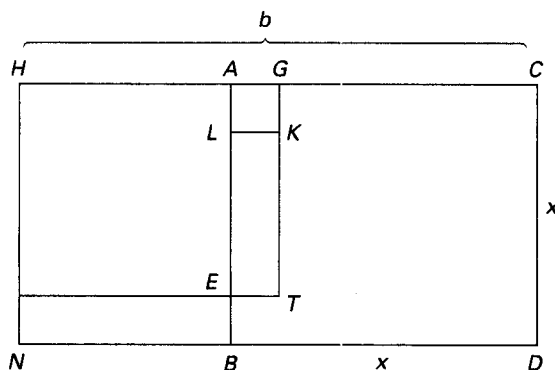


FIGURE 7.6
Ibn Turk’s geometric justification for one case of $x^2 + c = bx$.

Ibn Turk further noted that “there is the logical necessity of impossibility in this type of equation when the numerical quantity . . . is greater than [the square of] half the number of roots,”¹³ as, for example, in the case $x^2 + 30 = 10x$. Again, he resorted to a geometric argument. Assuming that G is located on the segment AH , we know as before that the square $KMNT$ is greater than the rectangle $HABN$ (Fig. 7.7). But the conditions of the problem show that the latter rectangle equals 30 while the former only equals 25. A similar argument works in the case where G is located on CA .

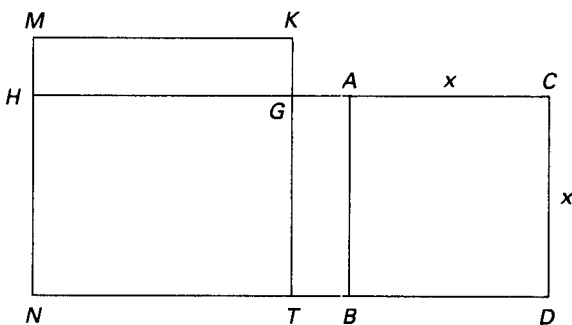


FIGURE 7.7
Ibn Turk’s geometric justification of the impossibility of solving $x^2 + 30 = 10x$.

Although the section on quadratic equations is the only part of ibn Turk’s algebra still extant, al-Khwārizmī’s text contains much else of interest, including an introduction to manipulation with algebraic expressions, explained by reference to similar manipulations with numbers. For example, he notes that if $a \pm b$ is multiplied by $c \pm d$, then four multiplications are necessary. Although none of his numbers are negative, he does know the rules for dealing with multiplication and signs. As he states, “If the units [b and d in our notation] . . . are positive, then the last multiplication is positive; if they are both negative,

then the fourth multiplication is likewise positive. But if one of them is positive and one negative, then the fourth multiplication is negative."¹⁴

Al-Khwārizmī's text continues with a large collection of problems, many of which involve these manipulations, and most of which result in a quadratic equation. For example, one problem states, "I have divided ten into two parts, and having multiplied each part by itself, I have put them together, and have added to them the difference of the two parts previously to their multiplication, and the amount of all this is fifty-four."¹⁵ It is not difficult to translate this problem into the equation $(10 - x)^2 + x^2 + (10 - x) - x = 54$. The author reduces this to the equation $x^2 + 28 = 11x$ and then uses his rule for this equation of type 5 to get $x = 4$. He ignores here the second root, $x = 7$, for then the sum of the two squares would be 58 and the conditions of the problem could not be met. In another example, al-Khwārizmī deals with a nonrational root: "I have divided ten into two parts; I have multiplied the one by ten and the other by itself, and the products were the same."¹⁶ The equation here is $10x = (10 - x)^2$ and the solution is $x = 15 - \sqrt{125}$. Here he again ignores the root with the positive sign, because $15 + \sqrt{125}$ could not be a "part" of 10.

Although al-Khwārizmī promised in his preface that he would write about what is "useful," very few of his problems leading to quadratic equations deal with any "practical" ideas. Many of them are similar to the previous examples and begin with "I have divided ten into two parts." There are a few problems concerned with dividing money among a certain number of men, but even these problems are in no sense practical. In fact, one of these reduces to the equation $x^2 + x = 3/4$, where x is the number of men, the solution of which is $x = 1/2$! An entire section of the text is devoted to elementary problems of mensuration, which will be discussed later, and a brief section is devoted to the "rule of three," but neither of these provides any practical uses of quadratic equations either. Finally, the second half of the text is devoted entirely to problems of inheritance. Dozens of complicated situations are presented, for the solution of which one needs to be familiar with Islamic legacy laws. The actual mathematics needed, however, is never more complicated than the solution of linear equations. One can only conclude that although al-Khwārizmī was interested in teaching his readers how to solve mathematical problems, and especially how to deal with quadratic equations, he could not think of any real-life situations that required these equations. Things apparently had not changed in this regard since the time of the Babylonians.

7.2.2 The Algebra of Thābit ibn Qurra and Abū Kāmil

Within 50 years of the works by al-Khwārizmī and ibn Turk, the Islamic mathematicians had decided that the necessary geometric foundations to the algebraic solution of quadratic equations should be based on the work of Euclid rather than on the ancient traditions. Perhaps the earliest of these justifications was given by Thābit ibn Qurra (c. 830–890). Thābit was born in Harran (now in southern Turkey), was discovered there by one of the scholars from the House of Wisdom, and was brought to Baghdad in about 870 where he himself became a great scholar. Among his many writings on mathematical topics is a short work entitled *Qawl fī taṣḥīḥ masā'il al-jabr bi l-barāhīn al-handasīya* (*On the Verification of Problems of Algebra by Geometrical Proofs*). To solve the equation $x^2 + bx = c$, for example, Thābit used Fig. 7.8, where AB represents x , square $ABCD$ represents x^2 , and BE represents b . It follows that the rectangle $DE = AB \times EA$ represents c . If W is the