ARGUING FROM REPUGNANCE: TRANSFER OF INTUITION IN MATHEMATICS AND PHILOSOPHY

Abstract. In analytic philosophy, how can we reconcile truth claims with the reliance on intuition to discover them? In this paper, we address this question by tracking a particular logical move, in which the repugnance of a consequence is used to reject a premise. We consider the nature and pitfalls of certification by shared intuition in mathematics and philosophy and discuss its transfer across logical implication. Finally, various responses of the mathematical community to its "crisis in intuition" are examined for their applicability to philosophy.

1. Introduction

Analytic philosophy has at its core a bundle of methods broadly construed as *analysis*: the taking apart, relating, and putting back together of ideas. In many instances, such as in conceptual analysis or in arguing toward reflective equilibrium, these methods ineliminably call on intuitions to make judgments about the deconstructed, reconstituted, or reassembled ideas. There has been a recent surge of attention paid to the legitimacy of these moves. The aim here is to scrutinize the reliance on intuition via juxtaposition of philosophical and mathematical arguments. I will focus on special classes of intuitions in this paper: intuitive knowledge of truth or falsity, intuitive understanding of concepts, and intuitive moral judgments.[1] In particular my main concern is a special form of appeal to intuition – a sort of soft *modus tollens* – that I will describe further below.

First let us consider the special place of intuition in modern philosophical argumentation. In some views, analytic philosophy uses intuition so crucially and intimately that in fact the whole of the field can be defined as "the effort to attain maximum clarity regarding the content and basis of our intuitions" [Reth, 8]. Timothy Williamson, who has engaged deeply with the problem of intuition, grounds the discussion by allowing that "intuitions are presented as our evidence in philosophy" [Will]. Framing the associated issues, he writes:

> 'Intuition' plays a major role in contemporary analytic philosophy's self-understanding. Yet there is no agreed or even popular account of how intuition works, no accepted explanation of the hoped-for correlation between our having an intuition that P and its being true that P. Since analytic philosophy prides itself on its rigour, this blank space in its foundations looks like a methodological scandal.

Ultimately, though, Williamson finds that intuitions are "just" a class of judgments, and distances himself from the so-called "judgment sceptics." George Bealer similarly endorses the view of intuitions as evidence in mathematics and philosophy: he argues that "omitting intuitions from one's body of evidence leads to epistemic self-defeat" without giving a clear account of its mechanisms for discovering truth [Beal, 2].

Many authors have similarly emphasized practical indispensability. Gary Gutting defines intuitions as "simply the rock-bottom beliefs [philosophers] find themselves forced to take as basic in their search for philosophical truth," and indeed recognizes "the fact that we have no alternative to beginning with our own de facto intuitions, even though they have no certification beyond our inability to get past them" [Reth, 6-7]. He is seconded by Ernest Sosa: "It is hard to avoid appeal to direct intuition sooner or later" [Sosa, 8]. But indispensability is a weak endorsement of the value of intuition as evidence. And beyond the lack of positive certification, intuition has other demonstrable problems. None of these authors' analyses, for instance, resolves the concern raised below.

Here is a sketch of the kind of philosophical argument I am concerned with. An assertion to be evaluated is intricately argued to have a specific logical (or quasi-logical) relationship to a second assertion; then a strong and nearly universal intuition affirming or rejecting the second assertion is invoked. Frequently, the author will argue that a principle has a collection of logical consequences which are intuitively unattractive,

[1] Because of this focus, I will not need to touch on the psychological and phenomenological literature on physical and sensory intuition.

undermining the desirability of the original principle. This follows the form of *modus tollens*, one of the standard and ancient rules of inference, which holds that if $P \implies Q$ and Q is false, then P must be false. Here, in the most stripped-down version of what I will call the appeal to repugnance: by way of assessing P, we argue that $P \implies Q$ and that we all believe that Q is unpleasant (or *seems* false), so P should be rejected. If *modus tollens* is also known as denying the consequent, then perhaps this move can be termed "disliking the consequent." (Or, to be more charitable, "doubting the consequent.")

The usefulness of this process hinges on the presumption that the benefits of intuition transfer across logical implication, in the forward direction for positive intuitions, and in the backward direction for negative intuitions. Prominent examples of this reasoning occur across philosophical subfields, and I will offer up a selection in Section 4. When we are dissatisfied with the results of this procedure, we may be inclined to critique the logical relationship of the statements, rather than the transfer of intuitive certification.

Many other critiques can be (and have been) leveled at the reliance on intuition to discover facts or morality; various authors (for instance, Stich and co-authors [Sti1, Sti2]) have cast doubt on its universality and have claimed that it is heavily culturally determined. Mine is a more logistical objection, though I do not give up the right to these critiques. My main argument will be that even if shared intuition were stable and independent of culture and historical moment, it would still have a problematic transfer across the operations of logic.

I will take it as so obvious as not to need demonstration that personal intuitions can be wrong and misleading. It takes more work, though perhaps not a great deal more, to show that strongly and widely held intuitions can be wrong and misleading as well. In fact, something more is evident from simple examples: that intuition is not invariant under logical equivalence. In other words, pairs of statements can be exhibited which are rigorously logically equivalent in an appropriate sense and such that strong intuitive belief held by virtually all people will hold one to be right, the other wrong.[2] This shows as a corollary that intuition affirming one statement does not in general bolster its consequences (because a statement and any equivalent form are consequences of each other); at the same time, intuition undermining the implications of a premise need not tarnish the premise.

To accomplish this, I will take examples from mathematics, philosophy's neighbor in what are sometimes called the *a priori* disciplines, where the logical interdependence of statements can be studied with somewhat greater rigor. Following that, I will look to examples in metaphysics, epistemology, and ethics to consider the extent of the parallels. A "crisis in intuition" caused a quiet revolution in mathematics in the twentieth century, and I will discuss some of fallout below, with attention to how various responses might have counterparts in analytic philosophy.

2. Analysis, Intuition, Repugnance

To set up the juxtaposition of mathematical intuition with philosophical intuition, let us briefly examine the development of analytic philosophy. The relevant notion of analysis has roots in the Greek *analusis*, a method developed in relation to geometry and philosophy, as discussed for instance by Aristotle in his *Analytics*. Euclid, the ancient Greek axiomatizer of geometry, set down in his *Elements* what became for millenia the authoritative model for the codification of mathematical content. The style is interestingly mixed between synthetic presentation (the prose gives deductions of theorems from precepts that are taken as given) and the prior analytic methods that it suggests (breaking down propositions into their constituent ingredients, as a means of discovering a deductive pathway in the 'forward' direction). Some 600 years later, bookending the Greek period, Pappus' writings still show an explicit tension between analysis and synthesis in mathematical method.

Michael Beaney traces these Greek notions through medieval Islamic to early Modern European logic and mathematics, where for instance the *Logique* of Port Royal maintains attention to the dual roles of analysis-synthesis. ("The art of arranging a series of thoughts properly, either for discovering the truth when we do not know it, or for proving to others what we already know, can generally be called method." [Bean]) Kant's influential delineation of analytic and synthetic in the 18th century shows the enduring concern with the distinction in his period; despite the distinctions and tensions, both of these notions of analysis and synthesis are coequal constituents of a modern analytic method.

[2] Of course, the notion of agreement of "virtually all people" must be defined and defended. I will speak more about universality of intuition below, but I think that the standard of universality of intuitions met by my mathematical statements is at least as high as that used by the scholars whose work I describe.

At the turn of the twentieth century, the project to ground mathematics on solid logical foundations was launched in earnest, especially by Frege and Russell, who at the same time can be said to have founded modern analytic philosophy. “What characterizes analytic philosophy as it was founded by Frege and Russell is the role played by logical analysis, which depended on the development of modern logic.” [Bean]

Over this same long timespan of the history and prehistory of analytic method, the principals display a great deal of ambivalence about reliance on intuition, sometimes making full-throated endorsements of its evident usefulness, and sometimes building sophisticated structures of rigor and argumentation to extirpate intuitions as much as possible. Euclid, whose work was intended to give a deconstruction of even eminently reasonable geometric propositions into basic building blocks, regarded the building blocks as themselves obvious as well as indispensable. At the same time, his work displays an effort to be parsimonious with his postulates (discussed in detail in the next section).

Descartes is credited with a shift away from axioms in the seventeenth century through his development of the so-called *analytic geometry*, which gets away from the deconstruction and deductive reconstruction in Euclidean method by making problems amenable to direct algebraic solution. Descartes himself had an interesting take on the primacy of intuition in *Discourse on Method* (1637): there, he pledged “to include nothing more in my judgements than what presented itself to my mind so clearly and so distinctly that I had no occasion to doubt it.”[3]

Modern mathematics is replete with examples of deeply counterintuitive ideas, proofs, and constructions that slowly win acceptance and become drivers of major new directions, from Cantor’s cardinal numbers to Gödel’s true and unprovable facts. I will argue that in a period starting in the nineteenth century and coming to a head in the 1930s, a “crisis of intuition” occurred in mathematics resulting in a series of discipline-wide, subterranean shifts in how intuition is regarded and managed, and indeed how the assertions of mathematics themselves are framed.

Next, let us consider the evolving meanings of “repugnance.” In its most current usage, repugnance denotes strong visceral distaste or revulsion. A much older meaning of repugnance is tantamount to contradiction or serious incompatibility: one idea may be repugnant to another. The OED finds this use (A is repugnant to B if contrary to, divergent from, or standing against B) in English since 1387, but counterparts are readily found earlier in other languages. Repugnance has special relevance for this discussion of intuition because of its historical role as sometimes identical to contradiction, but sometimes a bit more general or softer than outright contradiction. Ascertaining repugnance has often been regarded as accessible to intuition in a way that contradiction may require more rigor to establish. Let me briefly survey some historical understandings of repugnance with a suggestive collection of instances.

- Job Chapter 21: “How then do ye comfort me in vain, whereas your answer is shown to be repugnant to truth?”
- Aquinas writes in *Summa Theologica* (late 13th century): “that which implies being and non-being at the same time is repugnant to the idea of an absolutely possible thing, within the scope of the divine omnipotence”; in other words, God can manifest all and only those phenomena which do not entail contradictions. He invokes repugnance on multiple other occasions in his commentaries, often in relation to non-intuitive properties of the infinite. Thus, sometimes in his own voice and sometimes in the voice of his Objectors, we hear statements such as “the infinite is repugnant not only to nature, but likewise to grace”; or “intention looks to the end, to which infinite progress is repugnant.”
- In *Leviathan* (1651), Hobbes writes that the idea of a sovereign beholden to laws is “repugnant to the nature of a Commonwealth.”
- Justice Bradley, in a concurring opinion for the Supreme Court in the case of Bradwell v. Illinois (1873), writes “The harmony, not to say the identity, of interests and views which belong, or should belong, to the family institution is repugnant to the idea of a woman adopting a distinct and independent career from that of her husband.”
- Pope Paul VI in his important encyclical letter Humanae Vitae (1968) writes against contraception as diminishing the sexual act: “Hence to use this divine gift while depriving it, even if only partially, of its meaning and purpose, is equally repugnant to the nature of man and of woman, and is consequently in opposition to the plan of God and His holy will.”

[3]I do not claim that this view is uniformly held across his work; several other views of intuition are presented in other parts of his writings, including some where he sternly cautions against letting confidence substitute for proof.

3. Case studies from mathematics

3.1. **Flavors of mathematical intuition.** I want to distinguish between three broad kinds of mathematical intuition of interest here. Though this taxonomy is secondary to the main point about arguments from repugnance, it will help later to clarify the steady-state of intuition's role in "post-crisis" mathematics.

The first is that of the extension of a concept, or what gets to count as an instance of a type. For instance, sixteenth-century Italian mathematicians who were working on the solution of cubic (degree 3) polynomials had become aware of solutions involving $\sqrt{-1}$, and in fact needed to use these solutions as intermediate steps in their calculations. However, solutions involving $\sqrt{-1}$ were rejected as final values because these were evidently not "real" numbers. This is only one episode in the controversial career of the concept of *number*—even zero has has contested status as a number over time.

Other (often-discussed) examples of definitions with messy or controversial boundaries include the notions of a *polyhedron*, a *function*, or a *set*. The case of a polyhedron was famously taken up by Imre Lakatos in *Proofs and Refutations*. In it he tells the story of Euler's formula, which states at its simplest that the numbers of vertices, edges, and faces of a polyhedron are related by $V - E + F = 2$ (for instance, if the polyhedron is a cube we have $8-12+6 = 2$). Lakatos traces the history of this formula as it withstood being pelted with a collection of would-be counterexamples, like a donut-shape built out of rectangular faces, for which $V - E + F = 0$. These examples put pressure on the common-sense understanding of the content of the polyhedron concept, and historically the definition was repeatedly refined long before the statement of the theorem was revisited.[4] This kind of unappealing example, intended to probe the boundaries of mathematical understandings, crops up frequently in mathematics—these are "pathological" examples, or as Lakatos dubbed them, "monsters."

Solomon Feferman turns his attention to the problems and paradoxes of reliance on intuition, with special concern for counterintuitive examples, in *Mathematical Intuition vs. Mathematical Monsters*. Undeterred by monsters, he articulates the view that rigorous definitions are just formalizations of concept-intuitions, saying that "the arithmetized notion of curve must be treated as a model of an intuitive concept which itself isolates and describes in an idealized form certain aspects of experience." [Fef, 6] His bottom line is not threatened by monsters, since for him intuition has a proper role that is different from the straightforward discovery of truth: "[Intuition] is essential for motivation of notions and results and to guide one's conceptions via tacit or explicit analogies in the transfer from familiar grounds to unfamiliar terrain."[5]

A second variety of intuition is that of the truth or plausibility of a mathematical statement. When obvious statements are subjected to attempts at rigorous proof from basic constituent principles, two surprising outcomes are frequently observed: in one case, the seemingly simple statement requires an intensely difficult proof, and in the second case, it resists proof altogether. Feferman cites the canonical example of a theorem that is easy to state, "obviously" true, and quite hard to prove: the Jordan curve theorem (every closed curve in the plane that does not intersect itself divides the plane into two parts, an *inside* and an *outside*). A more central example from the history of math is that of the fundamental theorem of algebra (a polynomial of degree n has exactly n solutions, with multiplicity, over the complex numbers), which resisted proof even at the hands of Gauss because one of the crucial notions, continuity, was elusive.

A third type is the intuitive acceptability or existence of abstract objects or constructions. In math, there are many examples of objects whose existence seems offensive. One easily stated is the so-called *Gabriel's horn*, which is the surface of revolution obtained by rotating the graph of $y = 1/x$ about the x-axis—its shape is infinitely long and flaring, hence the name. If one follows the very basic methods of calculus, it is easily verified that Gabriel's horn encloses finite volume, but has infinite surface area. (A colorful analogy has been made to a wedding cake which, though you could eat it in a day, would take all eternity to frost.) One can see why observers in the early calculus period might get up in arms, and Thomas Hobbes does not disappoint, declaring: "To understand this for sense it is not required that a man should be a geometrician or a logician, but that he should be mad." Indeed, it seems obvious that such an object can not exist, but it

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[4] It is striking to consider the mutating definitions of *polyhedron* together with George Bealer's claim that "the relevant central terms of the a priori disciplines are semantically stable" (23). For him, the polygon is an explicit example of straightforward concept possession, while I regard it to be a prototype of a semantically messy concept.

[5] Feferman draws on several of the same examples from mathematical history that I will here, but has a distinct set of concerns, seeking to produce a rapprochement between what he calls common-sense intuition and set-theoretical intuition. He ends up arguing that intuition and its monsters are useful but that the dissonances they produce must be dealt with case by case.

is unclear how to eliminate it without giving up much of calculus, a sacrifice which seemed more palatable to Hobbes than it is likely to seem to us.

For another very brief example illustrating the contested role of intuition in mathematical ontology (that is, the third type of mathematical intuition discussed here), let us return to the concept arguably most basic to mathematics: that of a number. Our familiar numbers entered the canon in fits and starts, and not without controversy, from irrationals like $\sqrt{2}$ to zero and negative numbers to the complex numbers mentioned above. But a more radical campaign for number status was taken up by Georg Cantor in the late 19th century when he argued that even infinite sets could be considered to have well-defined sizes, so that one infinite set might meaningfully be said to be larger than another. In this way, he could assign a size, or *cardinality*, to any set, finite or infinite, and he could perform several of the operations of arithmetic on these cardinal numbers, such as addition and exponentiation. He thus felt very strongly that his cardinals should be granted membership among the numbers. This was extremely controversial and its cool reception by some of the senior mathematicians of the time, together with Cantor's increasingly paranoid disposition, combined to marginalize him from the mathematical community. Fascinatingly, though he campaigned vigorously for the mathematical reputation of infinitely large numbers, he was scandalized by the idea of rehabilitating the infinitely small in a similar way, attacking infinitesimals as a "Cholera-bacillus" in mathematics [Daub, 233].

So, to summarize, we have several flavors of intuition.

- **First Type:** Extension of a concept.
- **Second Type:** Truth or plausibility of a statement.
- **Third Type:** Existence of an object.

This discussion is intended to set the stage for a more extended consideration of how well we are guided by intuition in the cases below.

3.2. **The parallel postulate.** Euclid laid down five basic postulates which he took to be self-evidently true—that is, five founding principles which were concordant with his geometric intuition and which he could not express as consequences of simpler principles.

(1) A straight line segment can be drawn joining any two points.
(2) Any straight line segment can be extended indefinitely in a straight line.
(3) Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
(4) All right angles are congruent.
(5) Given any straight line and a point not on it, there exists one and only one straight line which passes through that point and never intersects the first line, no matter how far they are extended.[6]

Several assumptions, in fact, are embedded in this axiom system. It is immediate to see that the fifth postulate does not hold in (usual three-dimensional) space, because for a fixed line and a point not on it, there are infinitely many choices meeting the requirement. So the two-dimensionality of the geometry at hand is built in to the famous Fifth.

As it turns out, more is built in. Discoveries in the nineteenth century would show that the Fifth presumes that the ambient geometry is flat as well, while the other postulates are neutral with respect to curvature.[7]

On the way to the eventual embrace of alternate geometries (alternatives, that is, to Euclid's flat plane), the Fifth was the subject of much debate. Euclid himself seems to have been dissatisfied with the idea that it could not be stated to sound as simple and natural as the others; in fact, he avoided employing the parallel postulate in the first twenty-eight proofs in his Elements, which is suggestive of his holding it apart. He probably believed, and in subsequent Greek mathematics it was widely believed, that the Fifth could be proved from the others, meaning that it was actually redundant and could be removed in the name of elegance.

[6]In fact, this version of the Fifth is not the original one, but a restatement called Playfair's postulate or the parallel postulate which is more easily parsed. Many alternate restatements are possible, but this is the most frequent and classical. (Another, perhaps even more appealing to intuition, holds that a curve which is equidistant from a straight line must itself be a straight line. Yet another version says that every triangle has internal angles which add to 180 degrees.)

[7]This requires that the notion of a "straight line" be replaced with the more general idea of a geodesic. Then the first four postulates, suitably interpreted, make sense in model spaces of arbitrary constant curvature.

Euclid and his contemporaries never succeeded in establishing the logical relationship of the Fifth postulate to the first four. Greek geometry enjoyed a major rebirth of interest in the Arab world, where scholars resumed efforts to "prove" the parallel postulate in the ninth through thirteenth centuries [al-D, 31-42].

Later, a Jesuit priest named Girolamo Saccheri (1667-1733) devoted the bulk of his career to the same task, that of completing Euclid's project and showing that the Fifth follows from the acceptable four. (In his own words, "No one doubts the truth of this proposition..." [FG, 512].) I will focus attention on the well-known story of Saccheri's attempts, although he may have been anticipated by Arab mathematicians of several centuries earlier (notably Omar Khayyam). Saccheri employed the method of proof by contradiction: he assumed the first four postulates and the negation of the Fifth, then endeavored to find a logical contradiction. Had he found one, this would indeed have shown that the first four suffice to prove the Fifth. No contradiction, strictly speaking, emerged, though he did discover a number of statements which seemed impossible—he found, for instance, that rectangles could not exist in his axiom system.[8]

Having been scrupulous about extraneous assumptions throughout his work on the problem of parallels, carefully distinguishing facts of the system from merely "natural assumptions," Saccheri lapsed in the last year of his life. It would be a reasonable reading of the history to presume that he felt mounting panic at the prospect of non-resolution. Consequently he declared his findings (like the nonexistence of rectangles) to be "repugnant to the nature of the straight line," a bad enough sin when no contradiction is to be found. That is, the features of the non-Euclidean system violated his concept-extension intuitions (the first type above), and the working assumption was therefore to be rejected, validating the shared intuition of the dependence of the Fifth postulate on the others (second type). His book bore the trumpeting title *Euclid Freed of Every Flaw* (Euclides ab omni naevo vindicatus).

In this case, then, strong intuition of the repugnance of alternate systems led Saccheri to a now-provably-false conclusion: that the parallel postulate was a logical consequence of Euclid's other four. It was only in the next century, two millennia after Euclid, that four mathematicians more-or-less independently realized that alternatives to the parallel postulate would produce viable alternate geometries; that is, the Fifth is independent of the first four after all.[9] With this, the bizarre-seeming statements considered by Saccheri to be tantamount to logical contradiction became valid theorems of hyperbolic and spherical geometry.[10]

Saccheri's story is particularly interesting for the present discussion for two reasons. First, his appeal to repugnance is explicit. Second, the universality of his Euclidean intuition should be familiar to philosophical readers since it was vigorously defended by Kant as the very basis of geometric and spatial reasoning. Kant, in fact, wrote in the early nineteenth century that no other set of geometric intuitions could even be held by the mind, just a few dozen years before the multiple discovery of mathematical worlds whose exploration relies on just such alternative intuitions.

3.3. **The axiom of choice.** A second example comes from considering the once-hotly-debated Axiom of Choice.[11] Consider the following four statements.

Statement 1 (Axiom of Choice): For any collection of sets, it is possible to choose one representative from each set.

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[8]The axiom system obtained by assuming that a fixed line has no parallels through a fixed point is modeled by a sphere, where the role of *lines* is played by great circles (circles as large as the equator). Let me explain briefly how spherical geometry sustains some counterintuitive facts, like the non-existence of rectangles. No two *lines* are parallel on the sphere because any two great circles must intersect (in an antipodal pair of points, in fact). All triangles on the sphere have an angle sum of greater than 180 degrees—for instance, the triangle formed by the North Pole and any two points on the equator has two right angles and one arbitrary one. If, instead, the existence of several parallels had been taken as an axiom, then the model space (called the hyperbolic plane) would have all triangles containing less than 180 degrees. So if a *rectangle* is by definition a figure whose edges are *lines* and which has four right angles, then indeed no rectangles exist in either of these models (because a putative rectangle contains 360 degrees, but can be decomposed into two triangles).

[9]Lobachevsky, Riemann, Bolyai, and Gauss all have claims on independently arriving at the notion of a non-Euclidean geometry. Gauss in particular had the theory very well developed in his notes, but never published it, and the others caught up a few decades later. I think it is likely that Gauss, though he had the technology to launch the new theory, recognized it as problematically under-conceptualized.

[10]That is, the question was ultimately converted to one of the existence of consistent non-Euclidean systems, which could be ratified by third-type intuitions.

[11]In fact, this axiom shares some history with the parallel postulate: it was originally considered to be a clearly true statement, but ultimately discovered to be independent of the other axioms of set theory (the Zermelo-Fraenkel axioms, denoted ZF, which can be formulated as five postulates and two schema). This meant that the Choice principle could be either upheld or denied along with ZF, so that it deserves the status of an axiom itself.

Statement 2 (The Well-Ordering Principle): A notion of order can be introduced on any set such that any two elements are comparable, and any subset has a smallest element.

Statement 3 (Cardinal Successors, Strong Form): For every (possibly infinite) set A, there is another set B which is strictly larger and such that any set C which is larger than A is at least as large as B.

Statement 4 (The Banach-Tarski Paradox, Weak Form): It is possible to partition a solid ball into five sets, then act on the five pieces by a series of rigid motions (in fact, just rotations will do), such that after the motions there will be two solid balls reassembled, each of the same volume as the original ball.

The first three statements are provably, rigorously logically equivalent, and the fourth is actually weaker, meaning that it is implied by the others but does not require their full strength.[12] Yet they vary greatly with respect to how intuitively acceptable they seem. The first statement may seem intuitively unobjectionable, yet accepting it forces us to assent to the fourth—that a ball can be doubled simply by rearranging its pieces—which Saccheri might well have called repugnant. The third statement is often seen as plausible, while the second seems murkier, but much less believable. (In particular, it is already hard to imagine a well-ordering of the real numbers, under which, for instance, there would be a smallest positive real. Note that, unlike the rational numbers, the reals can't even be enumerated in a list.[13])

The Banach-Tarski paradox was published in 1924, following a tumultuous fifty years in mathematics of repeated events highlighting community failures of intuition: Weierstrass's continuous but nowhere differentiable function (1872), Cantor's cardinals (1870s-1880s), Hilbert's basis theorem (1888), Russell's paradox (1901). Then in 1931, Kurt Gödel published his enormously influential incompleteness theorems, shaking up the most basic and widely held ideas about the foundations of mathematics. Gödel's advisor Hans Hahn wrote a 1933 tract, "The Crisis in Intuition," focusing on the non-Euclidean, non-Archimedean, and fractal geometries that were coming into focus at the time, and going so far as to argue that mathematics had to be purged of intuition. "Because intuition turned out to be deceptive in so many instances," he wrote, "and because propositions that had been accounted true by intuition were repeatedly proved false by logic, mathematicians became more and more sceptical of the validity of intuition. They learned that it is unsafe to accept any mathematical proposition, much less to base any mathematical discipline on intuitive convictions. Thus a demand arose for the expulsion of intuition from mathematical reasoning, and for the complete formalization of mathematics."[14] In my view, this asceticism is excessive but the sense of crisis is warranted. However, mathematics has negotiated the crisis with some interesting moves in both epistemology and practice. I will discuss these further below, after reviewing some examples of philosophical arguments.

4. Case studies from philosophy

I hope it is clearly established that immediate or naive intuition is not invariant under logical equivalence in the mathematical examples above, and furthermore that there exist pairs of equivalent statements which excite strong and opposite beliefs. Let us turn to philosophy.

Below, I offer a selection of examples from philosophical work that fit the model of argument from repugnance. I make no claim that the conclusion in each example is wrong. The aim is merely to spotlight arguments following the pattern described above.

4.1. **Metaphysics: Strong beliefs about imaginary cases.** Consider first Derek Parfit's work in metaphysics, particularly his investigations into personal identity. His central tool is a thought experiment about teletransportation, namely about a machine called "The Scanner" which records the exact states of all of

[12]Work of Gödel and Cohen shows that the Axiom of Choice (AC) is independent of the Zermelo-Fraenkel axioms (ZF), which as noted above are the usual axioms of set theory. Over ZF, Statements 1-3 are pairwise logically equivalent. In particular, each implies any of the others.

[13]Ulrich Felgner said: "We believe that the ZF-axioms describe in a correct way our intuitive contemplations concerning the notion of set. The axiom of choice (AC) is intuitively not so clear as the other ZF-axioms are, but we have learned to use it because it seems to be indispensable in proving mathematical theorems. On the other hand, the (AC) has 'strange' consequences, such as 'every set can be well-ordered' and we are unable to 'imagine' a well-ordering of the set of real numbers." cited in [AC]

[14]On the other hand, Harvey Friedman has recently proposed a program of restating various mathematical axiom systems in popular terms until "the common person responds 'Oh yes, everybody believes that. Don't you?' " This is a clear appeal to shared intuition for justifying the adoption of a particular formal system. It is unclear what benefits Friedman feels will be conferred on theorems derived from these axioms.

our protagonist's cells, then transmits the information by radio, at the speed of light, to Mars, where another machine called "The Replicator" will then create from new matter an identical brain and body. We are asked to decide whether personal identity is maintained in the process. He subjects this scenario to a number of permutations: the original body on Earth is destroyed, given a debilitating cardiac condition, or left unmolested; the process is either completely reliable or unpredictably so; there is or is not a lover with a sentimental attachment to the protagonist; and so on. For many of these scenario variations, he claims that people will have near-universal intuitions, which Parfit calls "strong beliefs," about whether the double is still the same person who entered the scanner. Parfit registers Wittgenstein's general objection to the use of counterfactuals, as well as Quine's dismissal of "the method of science fiction" on the grounds that it invests excessive logical force in words. Parfit's rebuttal to Quine: "This criticism might be justified if, when considering such imagined cases, we had no reactions. But these cases arouse in most of us strong beliefs. And these are beliefs, not about our words, but about ourselves. ... Though our beliefs are revealed most clearly when we consider imaginary cases, these beliefs also cover actual cases, and our own lives" [Parf, 200].

To reformulate Parfit's claim, it holds that for a philosopher who seeks to understand when two things are personally identical (both "me"), sometimes real situations seem ambiguous or cloudy. In those cases, the philosopher should create idealized fictional scenarios ("imaginary cases") which elicit relatively unambiguous strong beliefs and then should carefully theorize their logical relationship to the unknown cases which were the true object of intended study. Furthermore, these beliefs are seen to be valuable precisely because they feel *strong* and they feel *real*. Parfit's argument is subtler, of course, than claiming that strongly held beliefs must be true; quite to the contrary, he takes great care in sorting these out into some that he considers "true," others "false."[15] However, this method amounts to using shared intuition (and a very light sprinkling of science) to critique other shared intuition under the operations of logic. After all, if we were to use Parfit's method to evaluate Cardinal Successors or the Well-Ordering Principle (Statements 2 and 3 above) – statements about which our intuitions are conflicted or unclear – the process would have a very different outcome depending on whether we proceeded to consider the attractive consequence of the statement in question (the Axiom of Choice) or the unattractive consequence (the Banach-Tarski paradox). This form of argumentation is now seen to be problematic when shared intuition is the primary mechanism for approval or rejection of premises. In particular, whether or not strong belief provides any meaningful data about teletransportation, it is totally unclear that such belief says anything at all about actual cases.

4.2. **Epistemology: Conceptual analysis.** In the theory of knowledge, the hallmark example of a simple and influential modern argument comes from Edmund Gettier's three-page paper from 1963. In it, he offered scenarios, now called *Gettier cases*, which undermined a dominant definition at the time: knowledge equals true, justified belief.

Here is a sample Gettier case (not one of the original ones, but clear and simple) regarding the barns of Fake Barn County. You are driving through a countryside where you see what appear to be barns all along the road. What you don't know is that in fact they are facades of barns, such as you would see on a movie set. You point to one of them as you drive by, and you say to your traveling companion, "Hey, look, a barn." By sheer coincidence, you are pointing to the only true barn in Fake Barn County. If we were to accept that knowledge equals true justified belief, then we would have to agree that you "know" that you are pointing to a barn: it is your belief, it is justified by what you see, and it is true.

However, most of us would not like to say that this amounts to knowledge, because of the combination of misunderstanding and coincidence, and in Gettier's original paper he simply states that, while these cases clearly provide true justified belief, "it is equally clear that Smith does not *know*" the proposition at hand. That is, our shared intuition of the content or extension of the *knowledge* concept is dispositive. Since the true justified belief account produces this intuitively unacceptable outcome, we must reject the account.

4.3. **Ethics: Moral intuition.** Ethics might seem to employ intuition to different effect, since after all its goals are to plumb moral permissibility, which is held by some not to be a matter of truth or falsity. However, the mainstream view among ethicists is that moral statements have truth values like other statements

[15]For instance, in the chapter called "How We Are Not What We Believe," Parfit launches an argument that "most of us have a false view about ourselves, and about our actual lives. If we come to see this view as false, this may make a difference to our lives" (217).

of purported fact.[16] The connection to mathematics is particularly apt in this popular version of moral epistemology. As Richmond Campbell says, “many philosophers have strongly resisted [moral naturalism] and have proposed that moral knowledge has its basis in non-natural aspects of the world that can be apprehended only through a faculty of moral intuition or reason that is independent of sense experience. Moral reality, so conceived, is posited as sui generis, reducible to neither the natural nor the supernatural and requiring a mode of apprehension comparable to mathematical intuition” [Cam].

The staple arguments of moral reasoning center on familiar hypotheticals: sinking ships and the opportunity to save some lives at the cost of others, say. Like Parfit’s “imaginary cases,” these are designed to elicit strong beliefs; here, moral right and wrong are to be determined.

Let us examine the argument structure in a particular piece of ethical reasoning. Alastair Norcross has used appeal to intuition as a persuasive tool in an extremely vivid way in his “Puppies, Pigs, and People: Eating Meat and Marginal Cases.” His paper centers on the following scenario: Fred has discovered that he must subject puppies to torture and brutal death in order that he may enjoy the taste of chocolate. Norcross proceeds to argue that Fred’s situation agrees with that of the meat-eater in enough essential moral features that anyone objecting to Fred’s puppy-torture must also repudiate factory farming of meat. First of all, he employs a level of detail which seems unnecessary to make the logical point but is clearly helpful in eliciting strong belief (telling us, for instance, of puppy urine and feces on Fred’s basement floor and that the torture includes slicing off puppies’ noses with a hot knife). The nature and presumed universality of the moral intuition is stated in emphatic terms:

> Clearly, we are horrified by Fred’s behavior, and unconvinced by his attempted justification. I expect near universal agreement with this claim (the exceptions being those who are either inhumanly callous or thinking ahead, and wish to avoid the following conclusion, to which such agreement commits them). No decent person would even contemplate torturing puppies merely to enhance a gustatory experience. [Norc, 3]

Having satisfied himself of the strong beliefs evinced by this scenario, he then needs to connect the imaginary case to the actual common practice of eating meat. He systematically considers possible arguments against the equivalence of puppy-torture and meat-eating, ultimately rejecting each. Norcross brooks no possibility of intuitive divergence with his central claims, saying for instance that, “If someone were to assert that puppyishness’ or simply being a puppy’ were ethically relevant, I could do no more than favor them with an incredulous stare.” (26) In summary, “I conclude that our intuitions that Fred’s behavior is morally impermissible are accurate. Furthermore, given that the behavior of those who knowingly support factory farming is morally indistinguishable, it follows that their behavior is also morally impermissible.” (25) I take “morally indistinguishable” to mean equivalent over logical and moral axioms and under the operations of logic.[17] Here again, we have intuition validated by other intuition, strung out across chains of logical and quasi-logical implication. Many skeptical readers would surely find fault with the conclusion that the scenarios are “morally indistinguishable,” but problems linger even if this equivalence is granted. Norcross’ argument structure is subject to the same challenges that I applied to Parfit above.

4.4. **Bioethics: “The Wisdom of Repugnance”.** Leon Kass has posited a principle he calls “the wisdom of repugnance,” which others have dubbed “the yuck factor.” He developed this notion in his writing around the ethics of cloning, calling on “the emotional expression of deep wisdom, beyond reason’s power fully to articulate it” in his impassioned arguments for instituting a permanent, international ban on the practice of human cloning.[18]

To universalize the recoil reaction, he cites a list of possible second- and third-order consequences of human cloning, like “mother-daughter twins” or “a woman giving birth to and rearing a genetic copy of... her deceased father.” Then, in a now-familiar pattern, Kass claims nearly universal strong belief: “ ‘Offensive.’ ‘Grotesque.’ ‘Revolting.’ ‘Repugnant.’ ‘Repulsive.’ These are the words most commonly heard regarding the prospect of human cloning. Such reactions come both from the man or woman in the street and from the intellectuals, from believers and atheists, from humanists and scientists.” He concedes that instinct and not logic grounds these reactions, saying, “We are repelled by the prospect of cloning human beings because

[16] This stands in contrast to other strands of ethics, like moral relativism, emotivism, and expressivism.

[17] For instance, something like the ZF axiom of replacement provides a means to get from arguments about puppies to arguments about cows.

[18] Ruth Macklin aptly counters, “While it is certainly true that repugnance may be the bearer of wisdom, it may also be the bearer of simple and thoughtless prejudice” [Mun, 719].

we intuit and feel, immediately and without argument, the violation of things that we rightfully hold dear." [Kass]

Like Parfit, Kass wants to sidestep the evident problems of reliance on intuition (acknowledging, as he does, that "Revulsion is not an argument") by claiming that his usage is special and limited: he identifies its role as evidence rather than argument and restricts its use to "crucial cases" with near-unanimity. But if there is substantive epistemological commonality between mathematical and moral intuition, then the examples above show that the problems outreach this attempted solution.

The parallel between Kass and Saccheri seems instructive. Each seeks to fall back on repugnance to damn an elusive opponent, and each is in fact justified in citing widespread agreement with his claims—Saccheri even more so than Kass, as the historical record shows some considerable consensus in intellectual circles of ancient Greece, medieval Islam, and Enlightenment Europe upholding his key intuition. Just one hundred and fifty years after Saccheri's work, his successors would find his framework to have been clearly limited, though, and Kass' case seems highly susceptible as well to the shifting of public wisdom with the passage of time. More to the point, they suffer from parallel structural problems.

5. Away from naive intuition

Some significant interest can come out of these battles. In particular, the trail of "monsters" often points the way to a new theory. An excellent example of this can be seen by returning to the naive statement of Euler's formula: $V - E + F = 2$ for all polyhedra. As discussed, historical objectors found a sequence of counterexamples to the simple principle, suggesting progressively more unwieldy definitions of polyhedron in order to exclude the exceptional cases. But for many, these jury-rigged definitions offended a guiding sense of aesthetic simplicity. From a modern perspective, the "right" theorem about $V - E + F$ is quite elegantly stated, in terms of a theory which had not been developed yet at that point: the point of view of *topology*, where rigid geometric objects are replaced by stretchable rubber-esque sheets. Thus the theorem states that $V - E + F$ is a *topological invariant*, so that it is equal to 2 for all polyhedra which are topologically equivalent to the sphere, and not for those equivalent to, say, the torus (or donut surface).

And indeed Cantor was wrong about the infinitesimals, which were theorized up to his standards of rigor in the 1960s by the so-called *non-standard analysis* of Abraham Robinson.

But these and other episodes look much less fraught in retrospect, and in fact it is hard for contemporary mathematicians to understand what was at stake at the time in these various controversies. This is because mathematics, as a field, has made three separate moves to address the crisis in intuition.

(1) Move to stipulative definitions and, when pressed at least, to a contingent epistemology;
(2) Build pedagogical systems to install refined intuitive systems special to subfields;
(3) Continually seek theories to assimilate pathologies.

Here, by stipulative definitions, I mean those which define an object by its properties. I mean to oppose this to earlier naturalized definitions, such as Euclid's classic "a point is that which has no part; a line is breadthless length." Upon this intuitive conceptual foundation, he then presents his five (irreducible, obvious) axioms, and begins construction. In a more modern treatment, we would call something a *Euclidean plane* exactly when it is a set which has subsets called points and lines and notions of betweenness and congruence satisfying a list of 17 axioms (I1-3,B1-4,C1-6,E, and P) [Hart]. (Here, P is the parallel postulate, and many of the others are careful recordings of Euclid's hidden assumptions.) Thus, what was formerly a statement of fact about intuitively natural objects is converted to a statement of compatibility of abstract objects with stipulated features. In this way, the first move sets up the second move, and the three kinds of mathematical intuition detailed above can be interlocked and exchanged.

These moves are by no means completely foreign to analytic philosophy.

A philosophical defense of the first move can be found in Sally Haslanger's *What Good Are Our Intuitions?*, where she argues that stipulative definitions are crucial to social constructionist goals in philosophy. She uses the formulation that these are *ameliorative*, as opposed to *conceptual* or *descriptive* projects—one seeks to understand what work a term is being asked to do, and to stipulate the best definition to meet those, often social, needs. [Hasl, 6-7] She considers terms like *parent* and *race*, and finds that definitions that "debunk our ordinary understandings and so inevitably violate our intuitions" may nonetheless be best suited to a robust philosophical analysis. For instance, if a *parent* is a person who behaves in certain ways or performs certain functions, then we may prefer to define a parent as a primary caregiver rather than an immediate progenitor. Therefore we must abandon the perspective that "the adequacy of a philosophical analysis is a matter of the degree to which it captures and organizes our intuitions" (5).

We have already seen simple appeals to repugnance in the work of Derek Parfit, and elsewhere in his work he flirts with the second move. Aptly enough, this is found in the problem in population ethics known as "The Repugnant Conclusion." This is the observation that many popular philosophical views of moral utility force assent to something that seems unacceptable: "For any possible population of at least ten billion people, all with a very high quality of life, there must be some much larger imaginable population whose existence, if other things are equal, would be better even though its members have lives that are barely worth living." [Parf, 388] Many authors have followed the blueprint of the argument from repugnance; as Clark Wolf writes, "the conclusion is taken to be sufficiently repugnant to our evaluative and moral intuitions that we are justified in rejecting the plausible seeming premises that led us to it" [RC, 61]. Parfit himself ends his discussion by endorsing a view he calls Perfectionism, which is adopted to avoid the Repugnant Conclusion, but along the way he does discuss the possibility of simply accepting RC. This possibility is taken up, indeed, by a long list of other authors, including Sikora, Anglin, Ng, Attfield, Ryberg, Norcross, Foton, and Tännsjö [RC, 236]. From here, the pivot to the second move ascribed to post-crisis mathematics would be as follows: accept the Repugnant Conclusion not as a true fact about the world, but as a fact (or theorem) of Utilitarian ethics, which becomes a branch of ethics much as hyperbolic geometry is a branch of geometry.

As to the third move, it is also quite powerful. For instance, the Banach-Tarski paradox is not a threat at all in modern mathematics, and on the contrary served as a motivating scenario for the development of a very active field of research inquiry. A group on which we can put an invariant mean which is an obstruction to geometric paradoxes is called *amenable* (so that we would say that three-space admits paradoxes because its group of isometries is non-amenable), and a quick search of the literature will verify that amenable groups are a popular topic indeed. That is, a phenomenon which started out being perceived as threateningly counter-intuitive was rehabilitated as value-neutral, and then spun off a new research area whose interest is now far removed from its initiating crisis. I am not immediately aware of a philosophical counterpart.

6. Conclusion

Analytic philosophers often rely heavily on a kind of intuition for affirming truth claims which is not apparently different in kind from mathematical intuition. I have given instances of lay intuition about mathematical statements (as well as the consensus intuition of trained mathematicians) varying widely under the passage from one statement to a logically equivalent second statement. This suggests an onus, for this style of analytic argument, of taking more care about the blend of logic and popular intuition, whether on one hand by clarifying exactly what benefits intuition does confer and how they transfer or on the other hand by relaxing the truth claims made with its aid to, say, the status of useful models (of moral codes, personal identity, and so on) rather than statements of fact about the same.

A possible objection to the case made in this paper would posit that the examples presented here are themselves pathological or at least extreme and that, in the norm, shared intuition is safe. This view holds that a few examples should not undermine the general reliance on intuition any more than occasional mirages should undermine our trust in vision. Timothy Williamson embraces the comparison of intuitions derived from counterfactuals to visual testimony, saying, "Notoriously, eye-witnesses often disagree fundamentally in their descriptions of recent events, but it would be foolish to conclude that perception is not a source of knowledge, or to dismiss all eye-witness reports." But this seems to me to avoid noticing a transformation in our assessment of the value of evidence: when the stakes are very high, as in a capital murder case, what we seem to see (eyewitness testimony) is most called into question, and requires corroboration by other means. That is, Kass has it quite backwards when he claims that intuition is most valid in "crucial cases"—it is precisely in cases of great importance where we can least trust this useful, but highly fallible, faculty. I submit that sensitive dependence on formulation is indeed the norm for intuitions about complex ideas. The examples given here, rather than exceptions to the rule, are a clue to the weaknesses in shared intuitions—they may seem universal, timeless, and sure, but in fact are as dependent on phrasing and context as on logical content.

Furthermore, mathematics is not reasonably sequestered as a hard case, where intuition is especially unreliable. On the contrary, it has often been cited within the philosophical mainstream as offering a special realm of clear thinking. To take just one example, Locke's arguments in "An Essay Concerning Human

Understanding" appeal to mathematics in just this way: "For it is as repugnant to the idea of senseless matter that it should put into itself sense, perception, and knowledge, as it is repugnant to the idea of a triangle that it should put into itself greater angles than two right ones." (529) The quotation calls attention, once again, to the trainability of intuition, for if Locke were a twentieth-century mathematics student, he would likely have learned not only to prove theorems, but to make good guesses in advance of proof, about other kinds of triangles as well.

Must we then abandon intuition? Surely not. I certainly take Solomon Feferman's point to heart, echoing the philosophical consensus cited in the introduction, that "intuitive knowledge or understanding is not simply separated from that obtained by more or less systematic reasoning—the two frequently go hand in hand, and neither is dispensable in practice" [Fef, 2]. Indeed, a mathematician is not forced to Hahn's position, stated earlier, that mathematics must be cleansed of intuition. Neither, clearly, must philosophy be—but more scrutiny is demanded in articulating the role intuition can play in combination with logic.

References

[al-D] Al-Daffa, Ali and John Stroyls, Studies in the Exact Sciences in Medieval Islam. Dhahran, Saudi Arabia: University of Petroleum and Minerals, 1984.

[Beal] Bealer, George. "A Theory of the A Priori." Pacific Philosophical Quarterly 81 (2000), 1–30.

[Bean] Beaney, Michael, "Analysis", The Stanford Encyclopedia of Philosophy (Winter 2008 Edition), Edward N. Zalta (ed.), URL=http://plato.stanford.edu/archives/win2008/entries/analysis/

[Cam] Campbell, Richmond, "Moral Epistemology," The Stanford Encyclopedia of Philosophy (Winter 2003 Edition), Edward N. Zalta (ed.), URL=http://plato.stanford.edu/archives/win2003/entries/moral-epistemology/

[Daub] Dauben, Joseph Warren. Georg Cantor: his mathematics and philosophy of the infinite. Princeton, 1990.

[Reth] DePaul, Michael R. and William Ramsey, eds. Rethinking Intuition. Rowman & Littlefield, 1998.

[Ebb] Ebbinghaus, H.-D., J. Flum, and W. Thomas, Mathematical Logic, 2nd ed. Springer- Verlag.

[FG] Fauvel, John, and Jeremy Gray. The History of Mathematics: A Reader. Macmillan/Open University, 1987.

[Fef] Feferman, Soloman, "Mathematical Intuition vs. Mathematical Monsters," Synthese 125 (2000), 317-332.

[Fri] Friedman, Harvey, "Reflection." Notes for lecture at the Stanford Center for Humanities, 12/3/03. URL=http://www.mathpreprints.com/math/Preprint/HarveyFriedman/20031218/1/

[Gett] Gettier, Edmund. 1963. "Is Justified True Belief Knowledge?", Analysis 23, pp. 121-123

[Hahn] Hahn, Hans, "The crisis in intuition," in Empiricism, Logic and Mathematics, 73-102. Reidel, Dordrecht, 1980.

[Hart] Hartshorne, Robin. Geometry: Euclid and beyond. Springer, 2000.

[Hasl] Haslanger, Sally, "What Good Are Our Intuitions?: Philosophical Analysis and Social Kinds." Proceedings of the Aristotelian Society Supplementary Volume, vol. 80, no. 1 (2006): 89-118.

[AC] Herrlich, Horst. Axiom of choice. Springer, 2006.

[Kass] Kass, Leon. "The Wisdom of Repugnance." The New Republic, 2 June 1997.

[Mun] Munson, Ronald. Intervention and Reflection: Basic Issues in Medical Ethics, 6th ed. Belmont: Wadsworth, 2000.

[Norc] Norcross, Alastair. "Puppies, Pigs, and People: Eating Meat and Marginal Cases." URL=http://www.ruf.rice.edu/∼norcross/Puppies.pdf

[Parf] Parfit, Derek. Reasons and Persons. Oxford: Clarendon, 1984.

[RC] Ryberg, Jesper and Torbjörn Tännsjö, The repugnant conclusion: essays on population ethics. Library of ethics and applied philosophy (15). Springer, 2004.

[Sti1] Stich, Stephen. "Reflective Equilibrium, Analytic Epistemology, and the Problem of Cognitive Diversity," Synthese 74 (1988): 391-413.

[Sti2] Nichols, Shaun, Stephen Stich, and Jonathan M. Weinberg. "Meta-skepticism: Meditations in Ethno-epistemology," in Steven Luper, The skeptics: contemporary essays. Ashgate, 2003.

[Sosa] Sosa, Ernest. "A Defense of the Use of Intuitions in Philosophy," in Stich and His Critics, Bishop and Murphy, eds. Blackwell Publishers, 2005.

[Will] Williamson, Timothy. The Philosophy of Philosophy. Wiley, 2007.