Problem Set 1 Math 111, History of Math, Duchin

This first homework assignment will test you on the basics that you will need to do the problem sets in this course. Those include *modular arithmetic, geometric series, the binomial theorem, logarithms, induction, proof by contradiction, the fundamental theorem of arithmetic, set notation, bijection, polynomial arithmetic,* and *complex arithmetic*, plus the willingness to look up and learn some unfamiliar mathematics.

Instructions: try to get away from formulas and use your solutions to show an understanding of *why* things work. In particular, show all work and write full sentences that make your work readable.

(1) Prove by induction that $n^2 + n$ is always even for $n \in \mathbb{N}$. Now prove it directly.

(2) Prove by contradiction that n^2 odd $\implies n$ odd. Now prove it directly.

(3) Using modular arithmetic, show that there is no integer n such that

$$3n + 1 = 15n^3 + 21n^2 + 3.$$

(4) Given that $\log_a(b) = y$, find $\log_c(b)$ in terms of y, a, and c.

(5) I'm thinking of a number between 1 and N. If you make a guess, I will tell you if it is high or low (or just right). How many guesses do you need to find the number? (Don't assume you have good luck.)

(6) Given that $A \subset B$, say for each of the following whether or not it must be true.
 (a) $A \times A \subset B \times B$
 (b) $A \times B \subset A \times B$
 (c) $A \times B \subset B \times A$
 (d) $\emptyset \subset B$

(7) Factor $x^3 + x^2 - 4x - 4$, finding all three integer roots.

(8) Do the following polynomial division problems:
 - $x^5 - 1$ divided by $x - 1$.
 - $3x^4 - 2x^3 + 5x^2 - x - 14$ divided by $x^2 + 2$.

(9) The points on the graph of the parabola $y = x^2$ form a subset $P \subset \mathbb{R}^2$. The points of the y-axis form a subset $Y \subset \mathbb{R}^2$. Give a bijection from P to Y.

(10) Give a bijection $f : O \to T$, where $O = \{\text{odd integers}\}$ and $T = \{\text{integer powers of two}\}$.

(11) What is the sum of $3 + \frac{3}{5} + \frac{3}{25} + \frac{3}{125} + \cdots$?

(12) What is the coefficient of xy^3 in $(2x + 3y)^4$?

(13) Show that $(a + b)^p \equiv a^p + b^p \pmod{p}$ *when p is prime* and explain why this is called "the freshman's dream." Explain what happens if p is replaced with a composite number n.

(14) Compute each of the following, putting your answer in the form $a + bi$ ($a, b \in \mathbb{R}$).
 - $\frac{3}{i} + 5i$
 - $(4 - i)^3$

- $z + \overline{z}$, where $z = 172 + \pi i$.

(15) For a polynomial f, if $f(z_0) = 0$, what is $f(\overline{z_0})$?

(16) Giganto walks up a flight of n stairs; at each step, she chooses whether or not to skip a step. How many possible ways are there for her to climb the steps?

(17) Show that the sum of all the odd numbers up to your age is a perfect square. Prove that this will be true for the rest of your life.

(18) Let S be a subset of $\{1, 2, \ldots, 9\}$ such that $|S| = 6$. Show that S contains elements a and b such that $a + b = 10$.

(19) How many different integers are divisors of 120? How many different integers are divisors of $m = p^a \cdot q^b$, where p and q are prime?

(20) Consider the recursive formula below:

$$\begin{aligned} L_1 &= L_0(1 + L_0) \\ L_2 &= L_0(1 + L_1) \\ &\vdots \\ L_{k+1} &= L_0(1 + L_k). \end{aligned}$$

Find two simple formulas that give good approximations to $L_{9959697884}$, one for when L_0 is large and one for when L_0 is small. What is the value of your approximation when $L_0 = \frac{1}{10}$? When $L_0 = 522$?