# MATH 180, PROBLEM SET 3

(1) (a) Prove that every quadratic irrational is of the form $a+b\sqrt{d}$ for $a, b \in \mathbb{Q}$ and $d \in \mathbb{Z}$ not a perfect square.
(b) Write down the **fan** for a rational number of your choice, compare it to the list of convergents, and find the rational number in the Farey graph.
(c) Find a quadratic irrational number equal to $[4, 1, 3, \overline{7, 1}]$.

(2) Find the continued fraction expansion for $\sqrt{n^2+1}$, where $n$ is a natural number. Your answer should be in terms of $n$. Do the same for $\sqrt{n^2+n}$.

(3) **The cost of an approximation.** Using a calculator, expand $\alpha = \sqrt[3]{2}$ by finding the first twenty partial quotients in its continued fraction. (Explain the steps that you used on your calculator.) Find the first ten convergents. The **error term** of a rational approximation to a number is just

$$\left|\alpha - \frac{p}{q}\right|,$$

measuring how far the rational number is off by. We're interested in whether those error terms are small relative to $q$. Define the **cost** of an approximation $p/q$ to a number $\alpha$ to be

$$q^2 \cdot \left|\alpha - \frac{p}{q}\right|.$$

For instance, the cost of $34/27$ as an approximation to $\sqrt[3]{2}$ is about .48. For each of the first ten convergents to $\alpha = \sqrt[3]{2}$, find its cost to two significant digits. If you didn't know anything about continued fractions, you would think that $12599/10000$ is a good approximation to $\sqrt[3]{2}$. Find its cost. See if you can find any rational number that is not a convergent but still has low cost.

(4) **Euler's rule.** Let us use the notation $\langle a_0, a_1, \ldots, a_k\rangle$ to mean the Euler product, defined as a sum as follows. The first term is the product of all of the numbers. The next terms are the product of all but two of the numbers, where successive pairs are omitted (first leave out $a_0$ and $a_1$, then leave out $a_1$ and $a_2$, and so on). The next terms are the product of all but four numbers which come in two successive pairs. We treat the "empty product" as the number 1.

$$\langle a, b, c\rangle = abc + c + a, \qquad \langle a, b, c, d\rangle = abcd + cd + ad + ab + 1,$$

$$\langle a, b, c, d, e\rangle = abcde + cde + ade + abe + abc + e + c + a.$$

(a) What is $\langle 1, 2, 3, 4, 5, 6\rangle$? What is $\langle 6, 5, 4, 3, 2, 1\rangle$? Must you always get the same answer when you reverse the order of the numbers? Explain.
(b) Compute $\langle 1, 1, \ldots, 1\rangle$ where the number of terms (the number of 1's) is 2,3,4,5, and 6. What is the pattern?

Euler's rule is the following equality:

$$[a_0, a_1, \dots, a_k] = \frac{\langle a_0, a_1, \dots, a_k \rangle}{\langle a_1, \dots, a_k \rangle}.$$

In other words, the rule expresses the recursively defined $p_k$ and $q_k$ (numerators and denominators of convergents) in terms of Euler products.

(c) Let $\alpha = [a_0, a_1, \dots, a_k]$ be a finite-depth continued fraction and $\beta = [a_k, \dots, a_1, a_0]$ be its reverse. Consider the convergents to $\alpha$ (call them $p_i/q_i$) and the convergents to $\beta$ (call them $p'_i/q'_i$). Is it possible for $\alpha$ and $\beta$ to be distinct numbers, but have a convergent in common? (For instance, can $\frac{p_5}{q_5} = \frac{p'_5}{q'_5}$?)

Extra Credit:

Do a short write-up of the concepts of cutting sequence and derived sequence for the straight-line flow on the square grid in the plane. Work out the examples of slopes 11/5 and 9/7 (modifying the "rules of the game," if necessary) so that the values of the sequences recover the partial quotients of the continued fraction expansions.