MATH 180, PROBLEM SET 4

Read through the texts for the first portion of the course: Angell's chapter on continued fractions and the three handouts (Parts 1-3) by Hatcher. (All linked from the website.) This should reinforce the material from lectures.

(1) For p/q a lowest-terms rational number, let $C(p/q)$ be the Ford circle at that number (that is, the circle in the upper half-plane that is tangent to the real axis at p/q and has diameter $1/q^2$).
  (a) Show that $C(1/2)$ is tangent to $C(0/1)$. What is the point of tangency?
  (b) Finish the calculation from class, showing that if $a/b < c/d$ and $b < d$, then
$$C(a/b) \text{ is tangent to } C(c/d) \iff ad - bc = -1.$$
  (c) True or False: for any two circles with any real radii that are tangent to the x-axis and to each other, there is exactly one mutually tangent circle (a circle that is tangent to the x-axis and both of the other circles). Explain.
  (d) Show that if the two circles in the previous question are Ford circles, then there is a mutually tangent circle which is also a Ford circle.
  (e) Give a systematic way to list all of the p/q such that $C(p/q)$ is tangent to $C(2/5)$.
  (f) Show that two Ford circles can't overlap in more than one point.

(2) Recall that the s-cost of an approximation is $s\text{-Cost}(\frac{p}{q}, \alpha) = q^s \cdot |\alpha - \frac{p}{q}|$. In class, I sketched a proof that
$$\ell = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000000} + \cdots + \frac{1}{10^{n!}} + \cdots$$
is a Liouville number (that is, there is some fraction that has s-cost less than one for every s). Another Liouville number is
$$L = [0, 10, 100, 10^6, \ldots, 10^{n!}, \ldots].$$
Here are some steps to prove it is Liouville.
  (a) Show that $a_{n+1} = a_n^{n+1}$. Show that the error term $|L - p_n/q_n|$ is less than $1/a_n^{n+1}$.
  (b) Show that $(a_1 a_2 \cdots a_n) < q_n < 10(a_1 a_2 \cdots a_n)$. Show that $q_n < a_n^2$.
  (c) Using the above, show that the error term is bounded by $1/q_n^k$ for some power k that goes to infinity as n goes to infinity. Conclude that L is Liouville, and thus transcendental.
  (d) Which is bigger, ℓ or L? Write down decimal expansions for each of them to twenty places.

(3) Groups and group actions.
  (a) Draw Cayley graphs for two different presentations of the Klein-four group $\{(0,0), (0,1), (1,0), (1,1)\}$.
  (b) Is the action of S^1 on $\mathbb{R}^2$ by rotations a transitive action? What are the orbits?

(c) Let $SL_2(\mathbb{R})$ act on the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by fractional linear transformation: $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).z = \frac{az+b}{cz+d}$. Verify that this is an action.

(d) What is the orbit of 0 under the FLT action?

Some definitions and explanations for this question:

- S^1 is the circle. One way to write down the circle as a subset of the complex plane $\mathbb{C}$ is as is as $\{e^{i\alpha} : \alpha \in \mathbb{R}\}$. Then the rotation action on the complex plane $\mathbb{C}$ is just multiplication by $e^{i\alpha}$, or in other words, the action on $\mathbb{R}^2$ is just rotating by angle α about the origin.
- A transitive action is one where any point can be sent to any other point. The action of G on X is transitive, in other words, if $\forall x, y \in X$, $\exists g \in G$ such that $g.x = y$.
- $SL_2(\mathbb{R})$ is the set of two-by-two matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with real entries and determinant one ($ad - bc = 1$). It is a group under matrix multiplication.
- $\hat{\mathbb{C}}$ is called the extended complex plane: it is just the complex plane plus an extra point called ∞. We'll talk more about this later, but for now you just need to know how to do arithmetic with infinity, so here are the rules:

$$a/0 = \infty \text{ if } a \neq 0$$

$$\frac{a\cdot\infty+b}{c\cdot\infty+d} = \frac{a}{c}$$