HYPERBOLIC GEOMETRY HANDOUT I
MATH 180

In this installment, we consider a particularly interesting group action on the extended complex plane.

Definition 1. The **Riemann sphere** $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the topological sphere that you get from stereographically projecting the unit sphere $S^2 \subset \mathbb{R}^3$ to the xy-plane and adding the north pole as a point at infinity. An explicit map for stereographic projection is $(x, y, z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$. The **upper half-plane** is

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} | y > 0\} \subset \hat{\mathbb{C}}.$$

Definition 2. For any field F, the **general linear group** $GL_2(F)$ is defined to be two-by-two matrices with entries in F that have non-zero determinant. The **special linear group** $SL_2(F)$ is the subgroup of $GL_2(F)$ where the matrices are required to have determinant one.

Definition 3. The **Möbius transformations** are the maps $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form $f(z) = \frac{az+b}{cz+d}$ for $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

Definition 4. In $\mathbb{C}$, an **open ball** is the set of points within a certain distance of a fixed point:

$$B_r(p) := \{q \in \mathbb{C} \mid d(p, q) < r\}.$$

In $\hat{\mathbb{C}}$, an open ball centered at infinity is a set

$$B_r(\infty) := \{q \in \mathbb{C} \mid d(q, 0) > R\} \cup \{\infty\},$$

where R is related to r by stereographic projection: $R = \sqrt{\frac{1+\sqrt{1-r^2}}{1-\sqrt{1-r^2}}}$.

A set S in the Riemann sphere $\hat{\mathbb{C}}$ is **open** if at every point, a sufficiently small open ball around the point is contained in S:

$$\forall p \in S, \ \exists \epsilon > 0 \text{ such that } B_\epsilon(p) \subset S.$$

For a map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, we say it is **continuous** if for all open balls $U \subset \hat{\mathbb{C}}$, the set $f^{-1}(U)$ is open as well.

A **homeomorphism** of $\hat{\mathbb{C}}$ is a map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that is bijective, continuous, and with continuous inverse. To check that f is a homeomorphism, it suffices to find its inverse function $f^{-1} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and check that both f and f^{-1} map open sets to open sets. For this, it suffices to check that both f and f^{-1} map open balls to open sets.

A **homothety** is a scaling map: $z \mapsto kz$ for $k \in \mathbb{R}$.

An **affine map** is a map of the form $f(z) = az + b$ for $a, b \in \mathbb{C}$.

Notice that the Möbius transformations are a group action of $GL_2(\mathbb{C})$ on $\hat{\mathbb{C}}$, via $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).z = \frac{az+b}{cz+d}$. (You proved on your homework that matrix multiplication does the same thing as composition of fractional linear transformations—the same proof works here.) When we write $A.S$ for a set $S \subset \hat{\mathbb{C}}$, we mean apply A to the whole set: $A.S := \{A.z | z \in S\}$.

Also notice that from this definition of $\hat{\mathbb{C}}$, we can define circles in $\hat{\mathbb{C}}$ to be images under stereographic projection of circles on the sphere. That means they are either (a) circles in $\mathbb{C}$ or (b) $L \cup \{\infty\}$, for L a line in $\mathbb{C}$.

Here are some facts about Möbius transformations and related geometry.

(1) $SL_2(\mathbb{R})$ also acts on $\hat{\mathbb{C}}$ by fractional linear transformations, and this preserves $\mathbb{H}$. (You proved this on your homework.)

(2) The group of all Möbius transformations is generated by the affine maps plus $J(z) = -1/z$. $J(z)$ is called **inversion in the unit circle**; it maps the unit circle to itself, and swaps the inside and the outside.

(3) Two matrices $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ correspond to the same Möbius transformation if and only if they are multiples of each other. In particular, the only matrices that induce the identity transformation are $\left(\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right) = kI$.

(4) $GL_2(\mathbb{C})$ acts triply-transitively on $\hat{\mathbb{C}}$: for all $x, y, z \in \hat{\mathbb{C}}$ that are distinct from each other and $x', y', z' \in \hat{\mathbb{C}}$ that are distinct from each other, there is some $A \in GL_2(\mathbb{C})$ such that $A.x = x'$, $A.y = y'$, $A.z = z'$. In fact, there is exactly one Möbius transformation that carries (x, y, z) to (x', y', z').

(5) Three distinct points in $\hat{\mathbb{C}}$ uniquely determine a circle.

(6) Möbius transformations are homeomorphisms of $\hat{\mathbb{C}}$.

(7) Möbius transformations map circles to circles: if C is a circle and $A \in GL_2(\mathbb{C})$, then $A.C$ is also a circle.

(8) Möbius transformations preserve angle: if two curves C and C' have tangent lines L and L' meeting at angle θ, then $A.C$ and $A.C'$ also have tangent lines meeting at angle θ. (Done in class.)

Proof of (2). Calculation! If $c = 0$, then the transformation is affine and we are done. If not, say $m(z) = \frac{az+b}{cz+d}$. But then

$$m(z) = \frac{(az+b)c}{(cz+d)c} = \frac{acz+bc}{c^2z+cd} = \frac{acz+ad-(ad-bc)}{c^2z+cd} = \frac{a}{c} - \frac{ad-bc}{c^2z+cd}.$$

But then if $f(z) = c^2z + cd$ and $J(z) = -1/z$ and $h(z) = (ad-bc)z + \frac{a}{c}$, we finally get $m(z) = h \circ J \circ f(z)$. □

Proof of (3). This is straightforward. Suppose $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).z = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right).z$ for all z. Then $\frac{az+b}{cz+d} = \frac{\alpha z+\beta}{\gamma z+\delta}$, so $a\gamma z^2 + b\gamma z + a\delta z + b\delta = \alpha c z^2 + \alpha d z + \beta c z + \beta d$. But two polynomials are only equal for all values of the variable if their coefficients are equal, so $a\gamma = \alpha c$, $b\gamma + a\delta = \alpha d + \beta c$, and $b\delta = \beta d$. But then $a/\alpha = c/\gamma$. Let that value be m, which means $a = m\alpha$, $c = m\gamma$. Likewise, $b/\beta = d/\delta = n$, so $b = n\beta$ and $d = n\delta$. But now substituting in to $b\gamma + a\delta = \alpha d + \beta c$, we get

$$\frac{bc}{m} + \frac{ad}{n} = \frac{ad}{m} + \frac{bc}{n},$$

thus $(ad-bc)/m = (ad-bc)/n$, which means $n = m$ and thus $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = m\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$. □

Proof of (4). First we'll show that we can find a Möbius transformation sending $(0, 1, \infty)$ to any triple (z, w, q) of distinct points in $\hat{\mathbb{C}}$. Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then $A.0 = b/d$, $A.1 = \frac{a+b}{c+d}$, and $A.\infty = a/c$. Now z and q can't both be ∞ since they are distinct. Assume without loss of generality that $q \neq \infty$. Then $a/c \neq \infty$, so

I can let $c = 1$. Then it follows from the given equations that $a = q$, $d = \frac{q-w}{w-z}$, and $b = z \cdot \frac{q-w}{w-z}$. So I have found an A as desired. Why is this unique? One way to see it is that if I chose any other value for c in the process above, the whole matrix would have changed by a multiple, and as we saw above, this would induce the same Mobius transformation. Another way to see it is to suppose that two different matrices A and B both send $(0, 1, \infty)$ to (z, w, q). In that case, $A^{-1}B$ fixes $(0, 1, \infty)$. But then $a/c = \infty \implies c = 0$, and $b/d = 0 \implies b = 0$, and $\frac{a+b}{c+d} = 1 \implies a = c$, so $A^{-1}B = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ is a multiple of the identity matrix, which means A and B differ by a multiple.

So now we know that there's a unique Möbius transformation mapping $(0, 1, \infty)$ to (z, w, q), and therefore also mapping any (z, w, q) to $(0, 1, \infty)$. Composing two such maps lets us map any triple to any other triple. □

Proof of (5). If one of the three points is ∞ or if all three points are collinear, then the circle is the line through the points in the plane and we are done. So assume without loss of generality that z, w, q are non-collinear points in $\mathbb{C}$.

First consider z and w. Let L be the line of all points equidistant from z and w (that is, the perpendicular bisector of the segment between z and w). For a point p on L, let $f(p) = d(p, z) - d(p, q)$. It is clear that $f(p)$ is a real number that changes continuously as p is moved along L; it is also clear that $f(p)$ is sometimes positive and sometimes negative. By the intermediate value theorem, there must be some point p_0 on L such that $f(p_0) = 0$. But then $d(p_0, z) = d(p_0, q)$, and since it is on L we also have $d(p_0, z) = d(p_0, w)$. So p_0 is equidistant from all three points, and therefore a Euclidean circle centered at p_0 goes through all three points.

Three distinct points $z, w, q \in \hat{\mathbb{C}}$ can not lie on more than one circle, because distinct circles intersect in at most two points. □

Proof of (6). Möbius transformations are clearly bijections, since they have inverses that are also maps from $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$. We need to see that Möbius transformations map open sets to open sets. It suffices to show that open balls are mapped to open sets by affine maps and by inversion in the circle. This takes four cases.

Case 1: Affine maps (compositions of rotation, homothety, and translation) clearly send open balls in $\mathbb{C}$ to other open balls in $\mathbb{C}$.

Case 2: Affine maps applied to $B_r(\infty)$. The ball is everything in $\mathbb{C}$ outside a Euclidean circle of radius R, plus the point at infinity. Affine maps preserve infinity: $f(\infty) = \infty$. They take Euclidean circles to Euclidean circles, so the image $f(B_r(\infty))$ is everything outside a new Euclidean circle plus the point at infinity. To see that this is an open set, the only tricky point is to check that there is some $B_\epsilon(\infty)$ contained in it. But $B_\epsilon(\infty)$ is everything outside of a certain circle, whose radius gets big as ϵ gets small. Simply choose ϵ so small that that circle completely contains the boundary of $f(B_r(\infty))$.

Case 3: J applied to open balls $B_r(p) \subset \mathbb{C}$.

If the open ball does not contain 0, then the image does not contain ∞ and J is clearly a bicontinuous map on $\mathbb{C}$ (by calculus).

If the open ball does contain 0, then there is some $B_s(0)$ inside $B_r(p)$ and it suffices to check that $J(B_s(0))$ is open. Now $J(0) = \infty$ and J clearly maps the circle of radius M about the origin to the circle of radius $1/M$ about the origin. Thus J maps $B_s(0)$ to the outside of some circle around the origin, plus the point at infinity. That is, $J(B_s(0)) = B_t(\infty)$ for an appropriate t.

Case 4: J applied to $B_r(\infty)$. By exactly similar reasoning, J maps $B_r(\infty)$ to $B_t(0)$ for an appropriate t.

This completes the proof that affine maps and $J(z)$ both take all open balls to open sets. □

Proof of (7). Affine maps $f(z) = az + b$ are the composition of a rotation, a homothety, and a translation. To be precise, if $a = re^{i\theta}$, then $f(z) = t_b \circ \rho_\theta \circ h(z)$, where $h(z) = rz$. This is the composition of three maps that evidently preserve circles, so $f(z)$ does as well.

Exercise 1. Recall that if $z = x + iy$, then its complex conjugate is $\overline{z} = x - iy$. By converting z to x and y coordinates, show that the set of solutions to the equation

$$\alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma = 0$$

is a circle for every $\alpha, \gamma \in \mathbb{R}$ and $\beta \in \mathbb{C}$. (It may be a point or the empty set if $\alpha\gamma \geq \beta\overline{\beta}$. It is a straight line when $\alpha = 0$.)

Exercise 2. Using the previous exercise, show that $J(z) = -1/z$ preserves circles. To do this, assume that z is on a circle (it is a solution to an equation as in the previous problem.) Now let $w = -1/z$ and find a similar equation satisfied by w.

This completes the proof of (7). □

Exercise 3. Show that $f(x + iy) = x + 2iy$, $(f(\infty) = \infty)$ is a homeomorphism of $\hat{\mathbb{C}}$ that does not preserve circles. (To show that it is a homeomorphism, mimic the proof above for Möbius transformations, changing what is necessary.)

Exercise 4. Find the fixed points of $\left(\begin{smallmatrix} 2 & 5 \\ 3 & -1 \end{smallmatrix}\right)$. Where does it map the unit circle? Where does it map the real line?