... up to the one of 17 feet; here something stopped him (or: here he stopped).

Heath [**3**, p. 132] writes

... up to seventeen square feet, 'at which point for some reason he stopped'.

The Greek word μεχρι meaning "up to" or "until" has more the sense of "just short of" [**5**, p. 1123] and suggests what I believe to be true: Theodorus generalized the Pythagorean even-odd concept and, as shown by the theorem, got stuck at 17. He did not have the algebra necessary to prove the theorem in its full generality and would have handled each integer separately. It is worth quoting van der Waerden again [**6**, p. 109]:

For the Pythagoreans, even and odd are not only the fundamental concepts of arithmetic, but indeed the basic principles of all nature.

And,

Plato always defines Arithmetica as 'the theory of the even and the odd'.

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A Multiplicative Metric

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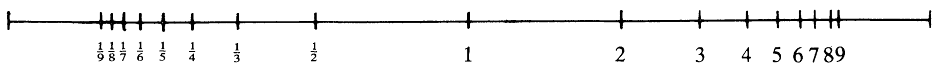
Collectors of metrics are a strange breed known to inhabit topological caves and caverns of analysis, ready to pounce on unwary examples. This specimen was hatched late at night while searching for metrics to spring on an elementary topology class.

For $x, y \in R$, define

$$d(x,y)=\begin{cases}\dfrac{|x-y|}{|x|+|y|}, & \text{if } x, y \text{ are not both } 0,\\ 0, & \text{if } x=y=0.\end{cases}$$

Clearly $d(x,y) \geqq 0$ and $d(x,y)=d(y,x)$ for all $x,y \in R$. The triangle inequality $d(x,y) \leqq d(x,z)+d(z,y)$ also holds, but its proof can be an algebraic nightmare. We give a version below which reduces calculations somewhat; perhaps a reader will be able to supply a short, neat proof.

First, note that the well-known inequality $|x-y| \leqq |x|+|y|$ implies that $d(x,y) \leqq 1$ for all x, y in R. It is easy to verify the following:



THE "MULTIPLICATIVE METRIC" VIEWED THROUGH EUCLIDEAN EYES: The point 1 serves as the center of the positive reals, and as $r \to \infty$, the distances $d(1/r, 1)$ and $d(1, r)$ remain equal to each other, increasing to 1 in value. Distance along the negative part of the real axis behaves in a similar fashion, but no distance is "added" if we cross the origin: -1, for instance, is no further away from $+1$ than is 0 since $d(-1, +1) = d(0, 1) = 1$.

(i) $d(0, x) = 1$ for all $x \neq 0$.
(ii) $d(x, y) = 1$ if x and y have opposite signs.
(iii) $0 < d(x, y) < 1$ if and only if $x \neq y$ and both are positive or both negative.
(iv) $d(x, y) = d(-x, -y)$.

To prove the triangle inequality, we consider separately three cases. If any two of the three real numbers x, y, z are equal, the inequality clearly holds. If any one of the three numbers x, y, z equals zero or differs in sign from the other two, then properties (i)–(iii) imply that $d(x, y) \leqq 1$ and $d(x, z) + d(z, y) > 1$. Finally, if the inequality were proved for x, y, z all positive, then property (iv) would imply that it also holds for x, y, z all negative, and we would be done.

So let x, y, z be all positive. The triangle inequality then holds if and only if the sum

$$\frac{|x-z|}{x+z} + \frac{|z-y|}{z+y} - \frac{|x-y|}{x+y}$$

is non-negative. Simplifying, this is equivalent to the following sum being non-negative: $y^2|x-z| + x^2|z-y| - z^2|x-y| + cd$, where $c = xz + yz + xy$ and $d = |x-z| + |z-y| - |x-y|$. Since interchanging x and y does not change the sum to be considered, computations to prove the sum is non-negative are necessary only for the three cases when $0 < x < y$. Verification of these three cases is left to the reader.

The metric d possesses both predictable and unexpected geometric properties. Since it is a bounded metric, it behaves predictably like a "perspective" metric. If we consider points equally spaced at intervals of length c along the real line R in the usual metric, these points seem to converge at $\pm\infty$ in the d-metric. In fact, for any $c \in R$, $\lim_{x\to+\infty} d(x, x+c) = \lim_{x\to-\infty} d(x, x+c) = 0$. It is also easy to verify the expected property that for any $c \in R$, $\lim_{x\to+\infty} d(x, c) = \lim_{x\to-\infty} d(x, c) = 1$. But we have already noted in (i), (ii) above that the d-distance from $x \neq 0$ on R to zero or to any other point having opposite sign is 1. Thus, for example, if c is a positive number, it is as "far" to zero as it is to the "ends" of the line R; in fact, it is no "farther" d-distance to any negative number than it is to zero.

Property (iv) states that the d-metric is invariant under the mapping which sends real numbers to their additive inverses; this property is shared by the usual metric on R. However, the d-metric is also invariant under the mapping which sends real numbers to their multiplicative inverses! It is easy to verify that $d(x, y) = d(1/x, 1/y)$ for all $x, y \neq 0$. An interesting consequence of this property is that for any $c \neq 0$, $d(c, 1) = d(1/c, 1)$ and $d(c, -1) = d(1/c, -1)$.

The metric d has yet another invariance property. The multiplicative group $G = R - \{0\}$ can be considered as a group of automorphisms of R, where each $a \in G$ is identified with the mapping $x \to ax$, $x \in R$. For each $a \in G$, and $x, y \in R$, it is easily shown that $d(ax, ay) = d(x, y)$; thus the metric d is invariant under the group G. An interesting comparison should be noted here. It is well known that the isometries of R with the usual metric are the translations $x \to x + a$, the reflection $x \to -x$, and the glide-reflections $x \to -x + a$. The multiplicative counterparts of these mappings, the dilations $x \to ax$, $a \neq 0$ the inversion $x \to 1/x$ and their composites $x \to a/x$, $a \neq 0$ are isometries of R with the d metric. (These last mappings are extended to all of R by sending $0 \to 0$.) It is an interesting exercise to verify that these are the only isometries of R with the d-metric. For this reason, we might label d a "multiplicative metric" for R.

The topology on R induced by the d-metric also contains some surprises, and reflects the invariance properties of d. For any point $x \in R$ we denote by $B(x; r)$ the open ball of radius r about x, namely the set $\{y \in R \mid d(x, y) < r\}$. If $r \geqq 1$, the ball $B(x; r)$ is all of R. For $0 < r < 1$, the ball about 0 is just the one-point set $\{0\}$, and for all $x \neq 0$, the sets $B(x; r)$ are open intervals in the usual topology on R. The precise description of these balls is

$$B(x;r) = \left\{ y \,\middle|\, x\frac{(1-r)}{(1+r)} < y < x\frac{(1+r)}{(1-r)} \right\}$$

if $x > 0$, and $B(x; r) = \{-y \mid y \in B(-x; r)\}$ if $x < 0$. Since $(1-r)/(1+r) = d(1, r)$, the ball $B(1; r)$ is just the set $\{y \mid d(1, r) < y < 1/d(1, r)\}$, and in general, if $x \neq 0$, $B(x; r) = \{xy \mid y \in B(1; r)\}$. Thus every open ball of radius $r < 1$ is a "multiplicative translate" of the open ball $B(1; r)$. (Compare this with the usual topology on R in which every open ball is an "additive translate" of an open ball about 0.) For example, $B(1; 1/2) = \{y \mid 1/3 < y < 3\}$; therefore, $B(1/3; 1/2) = (1/3)B(1; 1/2) = \{y \mid 1/9 < y < 1\}$, and $B(3/2; 1/2) = (3/2)B(1; 1/2) = \{y \mid 1/2 < y < 9/2\}$.

Since for $x \neq 0$, an open ball of radius r about x is an open interval in the usual topology on R, and the length of this interval approaches 0 as r approaches 0, the topology induced by d on $R - \{0\}$ is just the usual topology. The metric d disconnects the whole space R; in fact, R is the disjoint union of the three connected components R_+, $\{0\}$, R_-. No sequence can converge to 0 except the constant sequence $\{0\}$. However, a sequence converges to a limit $c \neq 0$ in the d-topology if and only if it converges to c in the usual topology on R.

The reader is invited to continue this investigation of the topological properties of the d-metric, and to attempt to extend it in a natural way to higher dimensions. Since the given proof of the triangle inequality depends on the ordering of R, it is not immediately apparent whether an analogous "multiplicative" metric can be defined on R^n.

Groups of Singular Matrices

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Groups of matrices under the operation of multiplication provide good examples in even the most elementary course of linear or abstract algebra. Since it is invariably assumed that the "identity" of such a group is the identity matrix (i.e., the diagonal matrix with all diagonal entries equal to 1), the only such groups studied are then necessarily composed of non-singular matrices.

What happens when we throw away this prejudicial assumption? Are there non-trivial groups of matrices containing singular matrices? If so, what do they look like? There are, as we show below, many examples of such groups. We will characterize all such groups, and discover (not surprisingly) that such groups are isomorphic to groups of non-singular matrices.

We begin our study by establishing notation and listing some elementary examples. All matrices considered in this note are square, of order n. The identity of a group of matrices under multiplication will be denoted as E, and the inverse of a matrix A in such a group will be denoted as A'; thus $AA' = A'A = E$. Only when A is non-singular will the usual notation A^{-1} for the inverse of A be