MATH 180, SOME SOLUTIONS

HW1

(1) Let's do one of the two inequalities.

$$\frac{a}{b} < \frac{a+c}{b+d} \iff a(b+d) < b(a+c) \iff ab+ad < ab+bc \iff ad < bc \iff \frac{a}{b} < \frac{c}{d}.$$

(3d) (cutting sequence for ϕ) The line of slope ϕ is the line $(t, \phi t)$, $t \in \mathbb{R}$. It crosses the vertical lines of the square grid when the x value is a whole number, which happens when $t \in \mathbb{Z}$. It crosses the horizontal lines of the square grid when the y value is a whole number, which happens when $\phi t \in \mathbb{Z}$, or in other words when $t = n/\phi$ for some whole number n. Let's call those times $t_0, t_1, t_2, \ldots$.

$t_0 = 0$	$t_1 = 1/\phi \approx 0.618$	$t_2 = 2/\phi \approx 1.236$	$t_3 = 3/\phi \approx 1.854$
$t_4 = 4/\phi \approx 2.472$	$t_5 = 5/\phi \approx 3.09$	$t_6 = 6/\phi \approx 3.708$	$t_7 = 7/\phi \approx 4.326$
$t_8 = 8/\phi \approx 4.944$	$t_9 = 9/\phi \approx 5.562$	$t_{10} = 10/\phi \approx 6.18$	$t_{11} = 11/\phi \approx 6.798$
$t_{12} = 12/\phi \approx 7.416$	$t_{13} = 13/\phi \approx 8.034$	$t_{14} = 14/\phi \approx 8.652$	$t_{15} = 15/\phi \approx 9.271$

That means

$$0 < \underbrace{t_1}_{b} < \underbrace{1}_{a} < \underbrace{t_2}_{b} < \underbrace{t_3}_{b} < \underbrace{2}_{a} < \underbrace{t_4}_{b} < \underbrace{3}_{a} < \underbrace{t_5}_{b} < \underbrace{t_6}_{b} < \cdots$$

$$\cdots < \underbrace{4}_{a} < \underbrace{t_7}_{b} < \underbrace{t_8}_{b} < \underbrace{5}_{a} < \underbrace{t_9}_{b} < \underbrace{6}_{a} < \underbrace{t_{10}}_{b} < \underbrace{t_{11}}_{b} < \underbrace{7}_{a} < \underbrace{t_{12}}_{b} < \cdots$$

$$\cdots < \underbrace{8}_{a} < \underbrace{t_{13}}_{b} < \underbrace{t_{14}}_{b} < \underbrace{9}_{a} < \underbrace{t_{15}}_{b} < \cdots$$

Now, you see an a in the cutting sequence when you cross a vertical line (whole numbers) and a b when you cross a horizontal line (t_i). So the cutting sequence begins $babbababbabbababbab\ldots$.

This is an "almost constant" sequence of value 1.

Now let $c = b$ and $d = ab$. Then the derived sequence begins $cdcddcdcddcd\ldots$. It has value 1.

Now let $e = cd$ and $f = d$. Then the next derived sequence begins $eefeefe\ldots$. It also has value 1.

If we continued this process as far as we like, we could see that we could continually keep passing to a derived sequence, and each of them has value 1. This is because the continued fraction expansion of ϕ is $[1, 1, 1, 1, \ldots]$.

HW2

(1) What is the relationship between the continued fraction expansion of p/q and that of q/p? The answer: one of them is $[0, a_1, a_2, \ldots, a_n]$ and the other is $[a_1, a_2, \ldots, a_n]$, where a_i are integers, $a_1 \neq 0$, and $n > 0$.

Proof: suppose p/q is less than one. Then $a_0 = 0$, since the first partial quotient is the integer part of p/q. But then $p/q = [0, a_1, \ldots, a_n]$. Now let $x = [a_1, \ldots, a_n]$. It is clear from the definition of the notation that $1/x = p/q$, so $x = q/p$.

On the other hand, suppose p/q is greater than or equal to one. Then $p/q = [b_0, b_1, \ldots, b_n]$ with $b_0 \geq 0$, again because it's the integer part of p/q. Now let $x = [0, b_0, b_1, \ldots, b_n]$. Again, it's clear that $1/(p/q) = x$, and therefore $x = q/p$.

(2) Let $x = [6, 4, 4, 4] = \frac{449}{72}$. Here's the table for figuring out the continued fraction info.

		6	4	4	4
0	1	6	25	106	449
1	0	1	4	17	72

This uses the "modified Fibonacci" rule

$$p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2}.$$

Here, the partial quotients are $6, 4, 4, 4$; the convergents are $6, \frac{25}{4}, \frac{106}{17}, \frac{449}{72}$. The complete quotients are the "tails" of the continued fraction, or what is left over from some a_k onward: in this case, they are $[4, 4, 4]$, $[4, 4]$, and $[4]$.

(3c) If p/q appears as a vertex in the Farey graph, show that q/p does as well.

We'll do induction on the number of times that the mediant operation was used.

For $0/1$ and $1/1$ and $1/0$, let's say we used the mediant 0 times.

Base case: using the mediant once. That is, $p/q = 1/2$ or $2/1$. Then it is clear that the reciprocal is obtained by using the mediant once, on the other side.

Inductive hypothesis: Suppose p/q is obtained by using the mediant at most n times. That is, there is a chain $m_0, m_1, m_2, \ldots, m_k = p/q$ for some $k \leq n$ such that each m_i is the mediant of m_{i-1} and some neighboring m_j for $j < i - 1$. The inductive hypothesis is that $\frac{1}{m_0}, \ldots, \frac{1}{m_k}$ is a mediant chain for q/p.

Inductive step: Suppose p/q has a mediant chain of length $n + 1$. Then p/q is the mediant of m_n with some m_j. Also, $1/m_n$ and $1/m_j$ are in the Farey graph, by the inductive hypothesis. But the mediant of a/b and c/d is the reciprocal of the mediant of b/a and d/c, so the mediant of $1/m_n$ and $1/m_j$, namely $1/(p/q) = q/p$, is in the Farey graph as well.

(4) The determinant of a matrix is the same as that of its transpose, and switching the rows or columns of a two-by-two matrix simply negates its determinant, so two rational numbers a/b and c/d are "special neighbors" if and only if $\det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \pm 1$.

Assume this is the case, so $ad - bc = \pm 1$. The mediant of a/b and c/d is $\frac{a+c}{b+d}$, so it suffices to show that this is a special neighbor of a/b.

$$\det\left(\begin{smallmatrix} a & b \\ a+c & b+d \end{smallmatrix}\right) = (ab+ad) - (ab+bc) = ad - bc = \pm 1,$$

as desired.

HW3

(4c) Can a continued fraction and its "reverse" have a convergent in common? Yes! Euler's rule tells you that a continued fraction and its reverse must have the same numerator but different denominators. Consider what it tells you, for instance, about the second convergents. Suppose $\alpha = [a_0, a_1, a_2, a_3, a_4, a_5]$. On one hand, $p_1/q_1 = \langle a_0, a_1 \rangle / a_1$, and on the other hand, its reverse yields $p'_1/q'_1 = \langle a_5, a_4 \rangle / a_4$. So if $a_0 = a_5$ and $a_1 = a_4$, they are the same.

For instance, consider $[4, 2, 5, 6, 2, 4]$. Its reverse is $[4, 2, 6, 5, 2, 4]$, and they are different numbers. However, $[4, 2]$ is a convergent for both of them.

HW4

(1c) For any two circles with real radii (in the upper half-plane, say) that are tangent to each other and the x-axis, must there be a mutually tangent circle which is also tangent to the x-axis? Yes! There is at least one and sometimes two; one has its point of tangency with the x-axis in between the other two, and there may be a second one which is bigger than the other two circles (for instance, $C(1)$ is tangent to both $C(1/2)$ and $C(2/3)$). (Note: if the two circles are one above and one below the axis, then the answer is of course no in general.)

Suppose C_1 has radius r_1 and is tangent to the x-axis at $x = x_1$; also C_2 has radius r_2 and is tangent to the x-axis at the point $x = x_2$, and say $x_1 < x_2$ without loss of generality. I will show that there is exactly one circle tangent to the real axis at $x_1 < x < x_2$ that is tangent to both C_1 and C_2.

First, fix any $x_1 < x < x_2$ and consider all the circles in the upper half-plane which are tangent to the x-axis at that point. Consider such a circle with radius R and call it $C(R, x)$. For extremely small R, this is disjoint from both C_1 and C_2, while for certain values of R, this intersects C_1 and C_2. Then there must exist R_1 such that $C(R_1, x)$ is tangent to C_1 and R_2 such that $C(R_2, x)$ is tangent to C_2, by continuity. Let $f(x) = R_2 - R_1$. Note that $f(x)$ is a continuous function and monotone decreasing. When x is very near x_1, we can see that $R_1 < R_2$, while when x is very near x_2, we have $R_2 > R_1$. Thus f is a continuous function on the interval (x_1, x_2) that changes sign; by the intermediate value theorem, there must be a (unique) c between x_1 and x_2 so that $f(c) = 0$. At this point, let $R_1 = R_2$, and the circle of that radius is tangent to both C_1 and C_2 by construction.

(1e) Which $C(p/q)$ are tangent to $C(2/5)$? In other words, what are all the neighbors of $2/5$ in the Farey graph?

Answer: neighbors in the Farey graph are exactly those rationals that appear as successive convergents of some continued fraction expansion. Now, $2/5 = [0,2,2] = [0,2,1,1]$. Therefore, $[0,2]$ and $[0,2,1]$ are neighbors (in other words, $1/2$ and $1/3$). Also, we can systematically list all the other neighbors:

$$[0,2,2,n] \text{ and } [0,2,1,1,n]$$

for natural numbers n. For instance, $[0,2,2,6] = 13/32$ is one of these neighbors.

(2) Prove that L is Liouville.

First, we observe that $a_{n+1} = 10^{(n+1)!} = (10^{n!})^{n+1} = a_n^{n+1}$. Since it is always true that the error term is less than $1/q_n q_{n+1}$, we have

$$|L - p_n/q_n| < \frac{1}{q_n q_{n+1}} < \frac{1}{q_{n+1}} = \frac{1}{a_{n+1}q_n + q_{n-1}} < \frac{1}{a_{n+1}} = \frac{1}{a_n^{n+1}}.$$

Now we show $(a_1 \cdots a_n) < q_n < 10(a_1 \cdots a_n)$.

$$(a_1 \cdots a_n) = 10^{1!+2!+\cdots+n!}$$

This number is one and a bunch of zeros: its number of digits is $1 + (1! + \cdots + n!)$ and it is therefore the smallest number with that many digits. On the other hand, it is easy to prove by induction that q_n has that same number of digits for $n > 1$. But $10(a_1 \cdots a_n)$ has one more digit, so that proves the inequality.

Next, we show that $q_n < a_n^2$. On one hand, $q_n < 10(a_1 \cdots a_n) = 10^{1+(1!+\cdots+n!)}$, and on the other hand, $a_n^2 = 10^{2 \cdot n!}$. So we need to show that

$$1 + (1! + \cdots + (n-1)!) < n! = n(n-1)! = \underbrace{(n-1)! + \cdots + (n-1)!}_{n \text{ times}}.$$

But this is true because the right-hand side and the left-hand side have the same number of terms and they can be compared term-by-term.

Note that $q_n < a_n^2 \implies q_n^{1/2} < a_n \implies \frac{1}{a_n} < \frac{1}{q_n^{1/2}}$.

But we know that

$$|L - p_n/q_n| < \frac{1}{a_n^{n+1}} < \frac{1}{q_n^{(n+1)/2}}.$$

But this exponent goes to infinity as n goes to infinity! So to find an approximation to L with s-cost less than 1, simply choose n large enough that $(n+1)/2$ is bigger than s.

HW5

(2b) For which α, β is $\langle \rho_\alpha, \rho_\beta \rangle$ a cyclic group? Answer: assuming neither α nor β is a multiple of 2π (that case is trivial), this is true if and only if there are representatives of the angles such that $\alpha/\beta \in \mathbb{Q}$. (For instance, rotation by $2\pi - 1$ and rotation by 3 radians do generate a cyclic group, because rotation by $2\pi - 1$ is the same as rotation by -1 radian, and that is a rational multiple of 3.)

Proof: Suppose there are representatives such that $\alpha/\beta = p/q$ in lowest terms. Let $G = \langle \rho_\alpha, \rho_\beta \rangle$ and let $C = \langle \rho_{\beta/q} \rangle$. This is a cyclic group, and I claim $G = C$. I need to show that β/q is an integer combination of α and β (this shows $C \leq G$), and that α and β are multiples of β/q (this shows $G \leq C$).

Since p, q are relatively prime, there exist integers m, n such that $mp + nq = 1$ (this is the Euclidean algorithm). But then $m(p/q) + n = 1/q$, and thus $m(p/q)\beta + n\beta = \beta/q$. This says $m\alpha + n\beta = \beta/q$, so I've shown that $C \leq G$.

On the other hand, β is clearly a multiple of β/q, while $\alpha = \beta(p/q) = p(\beta/q)$. So I've also shown that $G \leq C$.

In particular, this shows that for any two rational angles, they *always* generate a cyclic group!

To finish this off, I need to show that if α/β is irrational for all representatives of the angles, then they do not generate a cyclic group. If it did generate a cyclic group, then there would exist some γ such that $m\gamma = \alpha$ and $n\gamma = \beta$. Then $\alpha/\beta = m/n$ is rational, which is a contradiction.

HW 7

(3) We are given the map $f(x + iy) = x + 2iy$, $f(\infty) = \infty$, which stretches the complex plane by a factor of two in the horizontal direction. To show that it doesn't preserve circles, consider the points $\pm 1, \pm i$, which all lie on the unit circle. They are mapped to $\pm 1, \pm 2i$, which are the axial points on an ellipse and clearly do not lie on any common circle.

We must show that f is a homeomorphism. It is clearly a bijection, with inverse function $x + iy \mapsto x + iy/2$. To show that f and its inverse are continuous, we must show that they map open sets to open sets. It suffices to show that maps of the form $g(x + iy) = ax + biy$, which take circles to ellipses, map small open balls to open sets. (This is sufficient because any open set can be covered by open balls, by definition.) At every point in $\mathbb{C}$, this is clear, as the maps g are continuous functions of two variables. We just need to see that g maps an open neighborhood of ∞ to an open set.

We defined an open neighborhood of infinity to be some $B_r(\infty)$, which is everything strictly outside of the circle of radius R about 0 (see the handout). The map g takes this set to the outside of an ellipse with axes aR and bR. Assuming without loss of generality that $b > a$, that ellipse is completely contained in a circle of radius bR. But then $g(B_r(\infty))$ contains the set $B_s(\infty)$, where s is the value corresponding to the radius $S = bR$. Thus $g(B_r(\infty))$ contains an open neighborhood of infinity, so it is an open set.

Minitest 4

(1) The orbit of .8 is $0.8, 0.6, 0.2, 0.4, 0.8, \ldots$, which codes to $(1, 1, 0, 0, 1, 1, 0, 0, \ldots)$. Since $T(0.9) = 0.8$, its code is $(1, 1, 1, 0, 0, 1, 1, 0, 0, \ldots)$.

Something in between and purely periodic is, for instance, $x = (1, 1, 0, 1, 1, 0, \ldots)$. For this x, we have $T^3(x) = x$.

(2) To get the area of an octagon, break it down into triangles. If you connect one vertex to all the other vertices, you get six triangles, and their total angle adds up to the total angle of the octagon. Thus for an ideal octagon, you get six triangles with total angle zero. For an octagon with eight angles of $\pi/4$, you get six triangles with total angle 2π. The Gauss-Bonnet formula tells us that the area of a triangle is $\pi -$ (total angle), so the area of six triangles must be $6\pi -$ (total angle). Thus the ideal octagon has area 6π and the other one has total area 4π.

(3) We'll just see where each transformation takes 0 and i. Then since FLTs preserve circlehood and tangency, the image of the Ford circle $C = C(0/1)$ under A_j must be the circle tangent to the real line at $A_j.0$ and going through the point $A_j.i$.

$A_1.0 = \infty$, $A_1.i = i$ (so this one maps the circle C to the horizontal line through i.)

$A_2.0 = 7$, $A_2.i = i + 7$ (image circle is tangent to $\mathbb{R}$ at 7 and goes through $i + 7$—it's just C translated horizontally by 7 units.)

$A_3.0 = 3/2$, $A_3.i = \frac{5i+3}{3i+2}$ (image circle is tangent to $\mathbb{R}$ at $3/2$ and goes through $\frac{5i+3}{3i+2} = \frac{21+i}{13}$—it turns out to be the Ford circle $C(3/2)$, but that's a bit harder to prove.)