Self-similar groups: Where are the fractals?

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Perhaps we can recover the fractal associated with a complex dynamical system from its iterated monodromy group?

A basic example

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If $|w| < 1$, then w is drawn in to the critical point 0. If $|w| > 1$, then w spirals out towards ∞ (the other critical point).

The unit circle is the boundary between these two regions. So if $|w| = 1$, then there are pairs of points arbitrarily close to w that exhibit very different behavior under repeated application of the map $z \mapsto z^2$.

Julia sets

The **Julia set** $J(f)$ of a complex rational map f is (roughly speaking) the set of points in $\hat{\mathbb{C}}$ around which the dynamics of f are very sensitive to initial conditions. That is, close points will diverge greatly after a number of applications of f.

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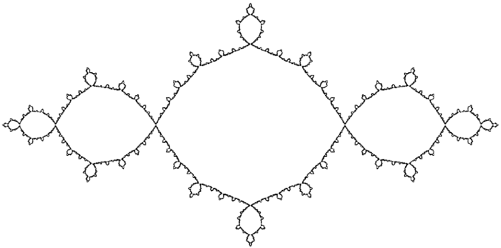
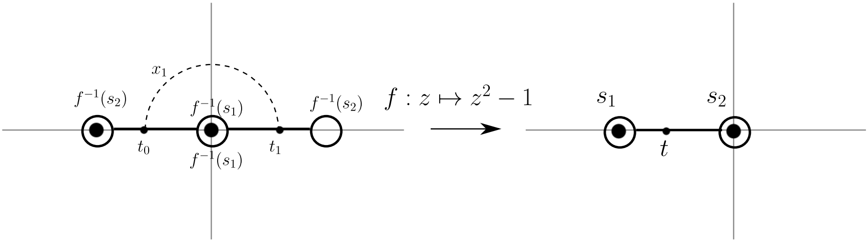
Julia sets of maps more complicated than $z \mapsto z^2$ are more complicated than the unit circle. Here is the Julia set of $z \mapsto z^2 - 1$:



The Basilica group: $IMG(z^2 - 1)$

The complex polynomial $f(z) = z^2 - 1$ is a double cover of $S^2 \setminus \{\infty, 0, -1\}$ by $S^2 \setminus \{\infty, 0, -1, 1\}$.



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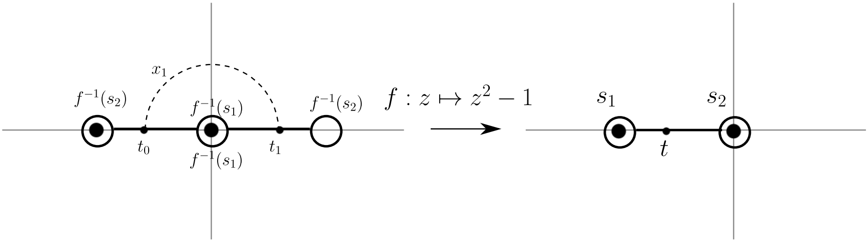

So we have the following actions of the generators:

$$s_1 = (01)(s_2, \mathtt{I}) \qquad s_2 = (s_1, \mathtt{I})$$

Schreier graphs

Definition

For G a group with finite generating set S acting on a space X, the (simplicial) **Schreier graph** of this action with respect to this generating set has vertex set equal to the points in X and an edge between vertices if and only if a generator takes one of the associated points to the other.

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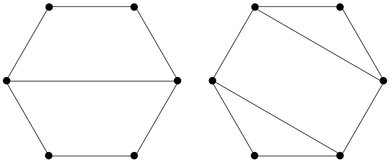
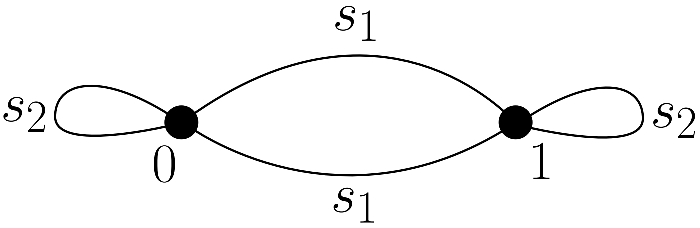
Examples: D_6 acting on the six vertices of a regular hexagon, but with different generating sets:



Schreier graphs of $IMG(z^2 - 1)$

This is the (not simplicial–sorry!) Schreier graph of $IMG(z^2 - 1)$ on the first level of the tree of preimages:



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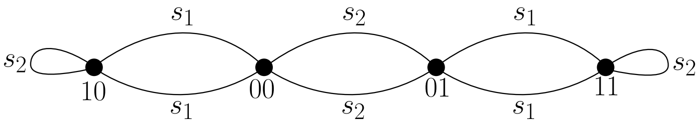

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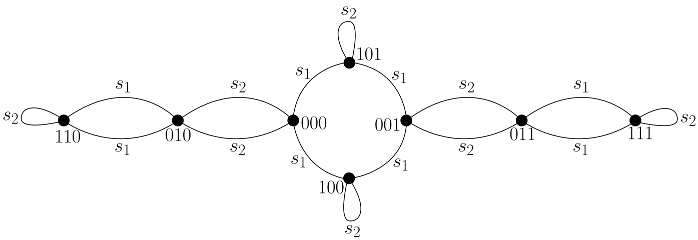
This is the (not simplicial–sorry!) Schreier graph of $IMG(z^2 - 1)$ on the *third* level of the tree of preimages:



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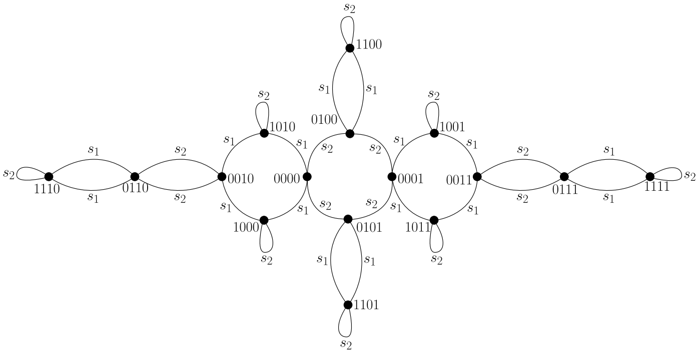

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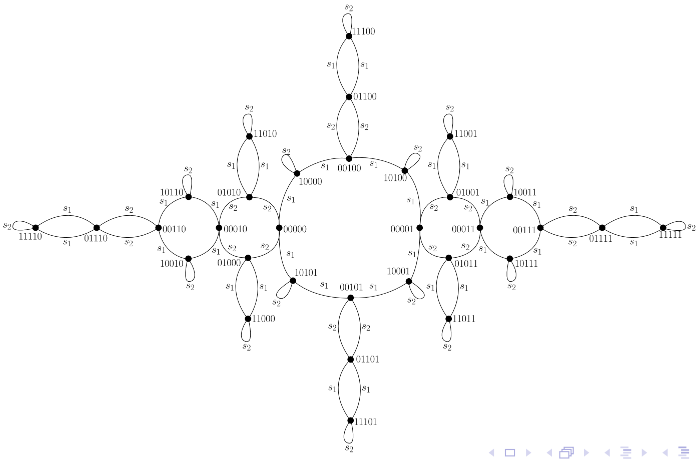
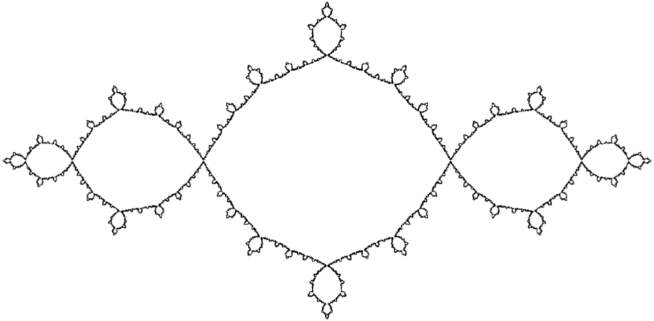


Schreier graphs of $IMG(z^2 - 1)$

This is the (not simplicial–sorry!) Schreier graph of $IMG(z^2 - 1)$ on the boundary of the tree of preimages:



Convergence of the Schreier graphs

Theorem (Nekrashevych)

The Schreier graphs of the action of a contracting self-similar group on the levels of the tree converge to a fractal set. In the case of an iterated monodromy group of a post-critically finite polynomial, this set is homeomorphic to the Julia set of the polynomial.

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Further, the shift map on the boundary of the tree gives us a dynamical system on the limit Schreier graph. This system is topologically conjugate to the action of the polynomial on its Julia set.

The Grigorchuk group

Let's compute the simplicial Schreier graphs of the action of the Grigorchuk group on the first few levels of the binary tree.

$$\begin{array}{llll} a(0w) = 1w & b(0w) = 0a(w) & c(0w) = 0a(w) & d(0w) = 0w \\ a(1w) = 0w & b(1w) = 1c(w) & c(1w) = 1d(w) & d(1w) = 1b(w) \end{array}$$

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- repeat...

Thus, every vertex at level $n \geq 2$ with a 0 occurring in the first $n-1$ places will have exactly 2 edges and the remaining two vertices ($1^{n-1}0$ and 1^n) will each have exactly one edge.

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So our limit fractal is a line segment.

The infinite dihedral group

The infinite dihedral group D_∞ admits a self-similar action:

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Hence, The Grigorchuk group and D_∞ have the same Schreier graphs, and thus the same limit fractal.

Unfortunately, this means that our groups $\rightarrow$ dynamical systems operation is not invertible.

References

- Laurent Bartholdi & Rostislav Grigorchuk, *On the spectrum of Hecke type operators related to some fractal groups*. Proc. Steklov Inst. Math. 231 (2000), no. 4, 1-41.
- Volodymyr Nekrashevych, *Self-similar groups*, Mathematical Surveys and Monographs, 117, Amer. Math. Soc., Providence, RI, 2005.