The Grigorchuk group:
An introduction to self-similar groups

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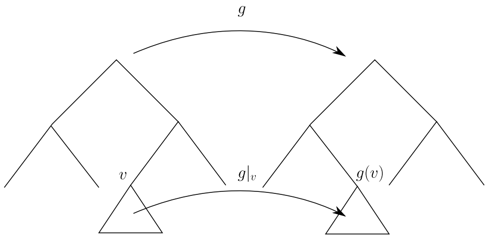
For $g \in Aut(T)$, we write $g|_v$ for the action of g from T_v to $T_{g(v)}$.



We call $g|_v \in Aut(T)$ the **restriction** of g at v

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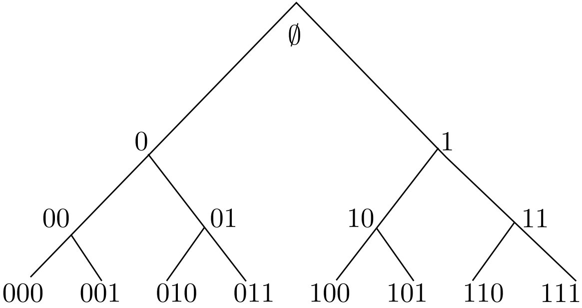
So how does b act on T (without referring to other generators)?

Another perspective

Or we could define $G = \langle a, b, c, d \rangle$ by identifying T with $\{0, 1\}^*$:

$$\begin{array}{llll} a(0w) = 1w & b(0w) = 0a(w) & c(0w) = 0a(w) & d(0w) = 0w \\ a(1w) = 0w & b(1w) = 1c(w) & c(1w) = 1d(w) & d(1w) = 1b(w) \end{array}$$



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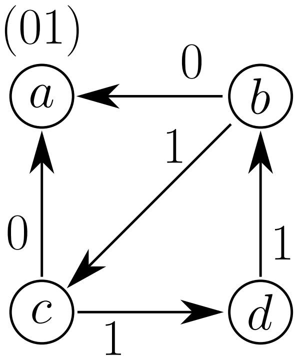

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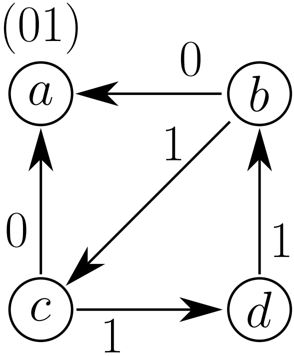

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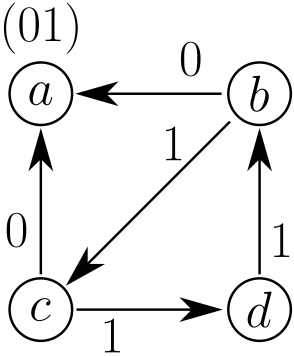

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Thus, $\{\mathtt{I}, b, c, d\}$ is isomorphic to the Klein four-group.

Reduced form

Thus every element $g \in G$ can be written in a **reduced form**:

$$g = s_0 \cdot a \cdot s_1 \cdot a \cdot s_2 \cdot a \cdot \ldots \cdot a \cdot s_k$$

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Aside: Can G be finite? No! The restriction map $g \mapsto g|_0$ from the stabilizer to G is surjective.

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In fact, for $\ell(g)$ the word length of g and $v \in \{0, 1\}$, we have that $\ell(g|_v) \leq \frac{1}{2}\ell(g) + \frac{1}{2}$. This is called **contraction**.

Applications of contraction

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2. Take the restrictions of g and repeat the previous step for $g|_0$ and $g|_1$. If both are trivial, then g is trivial. If either are not trivial, then g is not trivial. Otherwise, repeat.

Growth of a group

The **growth function** of a finitely-generated group H with respect to a finite generating set S is the function

$$\gamma(n) = \#\{h \in H \mid |h|_S \leq n\}$$

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Notice that this means that all groups grow at most exponentially.

Intermediate growth

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In practice, groups that are not virtually nilpotent all seemed to have exponential growth, leading Milnor to ask if there existed any groups with superpolynomial but subexponential growth rate.

The Grigorchuk group G is infinite torsion, and thus is not virtually nilpotent. So by Gromov it has superpolynomial growth.

Strong contraction

Grigorchuk showed that the following property implies subexponential growth:

If there exist constants $\lambda \in [0, 1)$, $p \in (0, 1]$, $C > 0$ such that the proportion of elements g in the stabilizer of the first level that have $\ell(g) \leq n$ satisfying

$$\ell(g|_0) + \ell(g|_1) \leq \lambda n + C$$

is at least p for all n, then G is **strongly contracting**.

Self-similar groups

Definition

A faithful action of a group G on a tree T by automorphisms is **self-similar** if for any vertex $v \in T$, then the action of $g|_v$ from T_v to $T_{g(v)}$ is equal to the action of some $h \in G$ when we identify T_v and $T_{g(v)}$ with the entire tree T.

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Self-similar groups also arise naturally in complex dynamics as **iterated monodromy groups**.

References

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