Exercises for Day 1

Lecture topics. $\mathbb{H}$, cross-ratio, $SL_2(\mathbb{Z}) \leq SL_2(\mathbb{R})$, trace classification, Farey graph, definitions of δ–hyperbolicity through insize, pairsums, and thin triangles.

Exercise 1.1. Hyperbolic plane basic exercises:

- Show that three points in $\hat{\mathbb{C}}$ uniquely determine a circle and that any two points in $\mathbb{H}$ lie on a unique orthocircle.
- Check that FLTs: are conformal, preserve $\mathbb{H}$, are compositions of affine maps and circle inversion, preserve hyperbolic length, preserve circles, act triply-transitively on $\hat{\mathbb{R}}$.
- Show that circles in the hyperbolic metric are round (Euclidean) circles in $\mathbb{H}$.
- Give a non-computational proof that every hyperbolic triangle contains a unique circle tangent to all three sides, and that the tangency points are the inpoints.

Exercise 1.2. Show that successive convergents of a continued fraction are adjacent in the Farey graph (example: $a + \frac{1}{b+1/c}$ and $a + \frac{1}{b+\frac{1}{c+1/d}}$). Since any real number has a continued fraction expansion given by the Euclidean algorithm, this means that there is a corresponding path in the Farey graph; if you draw it in $\mathbb{H}$, it converges back to the corresponding point on $\partial\mathbb{H} = \hat{\mathbb{R}}$. Draw the Farey path for 16/11 and for the golden ratio ϕ.

Exercise 1.3. Recall that the *ultrametric inequality* is $d(x,z) \leq \max\big(d(x,y),\ d(y,z)\big)$. Recall also that for four points x_1, x_2, x_3, x_4, a *pairsum* is $d(x_i,x_j) + d(x_k,x_l)$, where $\{i,j,k,l\} = \{1,2,3,4\}$. The pairsum definition of δ–hyperbolicity is that *for all four-tuples of points in the metric space, the two largest pairsums differ by no more than δ*. Prove that ultrametric spaces are 0–hyperbolic.

Exercise 1.4. For the upper half-plane model of $\mathbb{H}$ with $d(pi,qi) = \ln(|q/p|)$, show that the optimal hyperbolicity constant is $\delta = \ln(1+\sqrt{2})$ in the thin triangles definition, and $\delta = \ln\frac{3+\sqrt{5}}{2}$ in the insize definition (or make less precise argument that $\delta = 10$ suffices in each case!). Find a successful hyperbolicity constant in the 4–point definitions.

Exercise 1.5. Suppose a path segment γ in a geodesic δ–hyperbolic space has the property that any subsegment of length 8δ is geodesic. (We can call this path a *local geodesic*.) Let β be any geodesic between the endpoints of γ. Show that every point on γ is within 2δ of some point on β.

Exercise 1.6. *Similarities* are maps $\delta_n : X \to X$ with $d(\delta_n(x), \delta_n(y)) = n \cdot d(x,y)$.

(a) Under what conditions can a metric space with similarities $\{\delta_n\}_{n\in\mathbb{N}}$ be δ–hyperbolic?

(b) Now suppose that a metric space has almost-similarities, meaning a family of maps for $n \in \mathbb{N}$ for which the ratio

$$\frac{d(\delta_n(x), \delta_n(y))}{n \cdot d(x,y)} \to 1$$

for all $x \neq y$. Prove that almost-similarities suffice to provide obstructions to δ–hyperbolicity.

(c) Consider the maps $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\delta_n} \begin{pmatrix} 1 & nx & n^2z \\ 0 & 1 & ny \\ 0 & 0 & 1 \end{pmatrix}$ in the Heisenberg group $H(\mathbb{Z}) = \langle \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \rangle$. These are almost-similarities of the word metric. Find an estimate comparing $|g|$ with $|\delta_n g|$ (the lengths in the word metric) and use it to conclude that $H(\mathbb{Z})$ is not a hyperbolic group.

Exercises for Day 2

Lecture topics. QI and QI invariants, quasi-geodesics, groups and growth

Exercise 2.1. Let $C_2 = \mathsf{Cay}(\mathbb{Z}, \{2^n\}_{n\geq 0})$ and $C_3 = \mathsf{Cay}(\mathbb{Z}, \{3^n\}_{n\geq 0})$ be Cayley graphs with countably infinite generating sets. Fun open problem: is $C_2 \underset{QI}{\sim} C_3$? Here are some approachable exercises.

- Show that both graphs have infinite geodesics, and in fact that there are isometric embeddings of $\mathbb{Z}^d$ for every d into each of these graphs. (Consequence: both have infinite asymptotic dimension!)
- Show that they are not hyperbolic; each is one-ended; and each has linear divergence, if you know what that is.
- Show that reading a binary integer as a base-three integer by converting 1 digits to 2s (e.g., $1011_2 \mapsto 2022_3$) gives a QI embedding $C_2 \to C_3$, but that it is not coarsely surjective.
- For number theory buffs: show that the identity map $\mathbb{Z} \to \mathbb{Z}$ is not a QI from $C_2 \to C_3$.

Exercise 2.2. Recall that a Hamiltonian cycle in a graph visits every vertex once.

(a) Show that a finite rectangular $m \times n$ grid of points (with usual vertical and horizontal edges) has a Hamiltonian cycle except if m, n both odd.

(b) Show that $\mathsf{Cay}(\mathbb{Z}_m \times \mathbb{Z}_n,\ \pm\{(1,0),(0,1)\})$ has a Hamiltonian cycle, for any parity of m and n.

(c) Using the classification of abelian groups, show that all finite abelian groups have Cayley graphs with Hamiltonian cycles. Do all Cayley graphs of finite abelian groups have Hamiltonian cycles? (not the same question!)

Exercise 2.3. Compute $\beta_G^S(n)$ for $G = \mathbb{Z}^2$ with the standard generators and with the non-standard generators $S_{\mathsf{hex}} = \pm\{(1,0),(0,1),(1,1)\}$. Conclude that the growth function itself is not QI invariant (rather, only its rate is invariant). Nonetheless, for each S, there is a well-defined leading coefficient of growth given by $A = \lim\limits_{n\to\infty} \frac{\beta(n)}{n^2}$. Make a conjecture about how A depends on S.

Exercise 2.4. Prove that $SL_n(\mathbb{Z})$ has exponential growth for all $n \geq 2$, but is not hyperbolic for $n > 2$. Prove that Diestel-Leader graphs $\mathrm{DL}(m,n)$ have exponential growth but are not hyperbolic.

Exercise 2.5. Consider the spiral $t \cdot e^{i\cdot f(t)}$, which is parametrized by distance from the origin. It is an exercise in Bridson–Haefliger to show that this is a Euclidean quasi-geodesic for $f(t) = \ln(1+t)$. Do this meta-exercise: for what differentiable functions $f(t)$ is this the case?

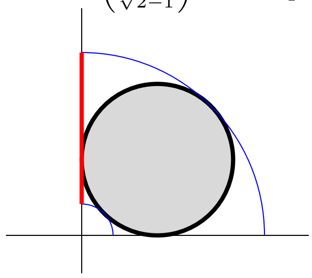
Exercise 2.6. Revisit the exercise from yesterday that 8δ–local geodesics are 2δ–close to true geodesics in a geodesic δ–hyperbolic space; (a) upgrade it to a proof that such local geodesics are in fact quasigeodesic. Come up with an example of such a local geodesic that is not geodesic (b) in $\mathbb{R}^2$ with the L^1 metric, and (c) in a hyperbolic space.

Exercises for Day 3

Lecture topics. Fellow-traveling, definitions of δ–hyperbolicity through divergence and contraction, algorithmic problems, Dehn's algorithm.

Exercise 3.1. Hyperbolic geodesic spaces have strong contraction: *there exists $M > 0$ such that for any geodesic γ, and any ball $B_r(x)$ of arbitrary radius that is disjoint from γ, the closest-point projection of the ball to the geodesic has diameter $\leq M$.*

(a) Using the figure, prove that $M = \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$ is the optimal contraction constant for $\mathbb{H}$.



Alternate statement of strong contraction: *there exists $M > 0$ so that if any geodesics α, β satisfy $\beta \cap \mathcal{N}_{2\delta}(\alpha) = \emptyset$, then the closest-point projection $\mathrm{proj}_\alpha(\beta)$ has diameter $\leq M$.*

(b) Prove directly that $\mathbb{H}$ satisfies this property, with a similar argument to above.

(c) Show that all δ–hyperbolic spaces do, say with $M = 5\delta$. (Not sharp!)

Exercise 3.2. Let's check that $PSL_2(\mathbb{Z}) = \langle T, J\rangle$ for $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. We will show that any $M \in \Gamma = PSL_2(\mathbb{Z})$ has a spelling in T and J by considering the action on $\mathbb{H}$.

(a) Show that the stabilizer of ∞ in Γ is $\langle T\rangle$ and that $\Gamma.\infty = \hat{\mathbb{Q}}$.

(b) Compute $T^a J T^{-b} J T^c J.\infty$. Show that $\langle T, J\rangle.\infty = \Gamma.\infty$.

(c) Show that there is some $A \in \langle T, J\rangle$ so that $A^{-1}M$ fixes infinity. Conclude that $M \in \langle T, J\rangle$.

(d) Explain why $SL_2(\mathbb{Z})$ is generated by the same two matrices.

Exercise 3.3. So we have $PSL_2(\mathbb{Z}) = \langle J, T \mid J^2 = I,\ (JT)^3 = I\rangle$. But this means we could modify the presentation by defining $B = JT$ and taking $PSL_2(\mathbb{Z}) = \langle J, B \mid J^2 = B^3 = I\rangle$.

Draw the Cayley graph and compute δ in the thin triangles definition of hyperbolicity. Give constants for Coornaert's definite growth ($C_1\alpha^n \leq \beta(n) \leq C_2\alpha^n$). Give a Dehn presentation.

Exercise 3.4. Yesterday you found growth functions for the standard generators and for the nonstandard genset S_{hex}. Show that the corresponding growth series are rational, actually computing the polynomials p, q for which $F(x) = p(x)/q(x)$ in each case.

Exercise 3.5. Recall that $\mathrm{asdim}(X) \leq n$ iff for any separation constant d there are $n+1$ families of d-separated, uniformly bounded open sets that together cover X. Modify the "bricks" proof that $\mathrm{asdim}(\mathbb{R}^2) = 2$ to show that $\mathrm{asdim}(\mathbb{H}) = 2$.

Exercise 3.6. The Heisenberg group $H(\mathbb{Z})$ is QI to $\mathbb{R}^3$ with an unusual metric in which $\delta_n(x, y, z) = (nx, ny, n^2 z)$ is a similarity, expanding distances by a factor of n. Conclude that $H(\mathbb{Z})$ has linear divergence.

EXERCISES FOR DAY 4

Lecture topics. Isoperimetric problems and Dehn functions, Gromov's gap, Wenger's sharp bound, boundaries at infinity, horofunctions.

Exercise 4.1. (a) The Heisenberg group $H(\mathbb{Z})$ has the presentation $\langle a, b, c : [a, b] = c,\ [a, c] = e,\ [b, c] = e\rangle$. Show that $a^\alpha b^\beta c^\gamma$ is a normal form, i.e., every group element has a unique representation in this form. Note that the presentation 2–complex is made up out of squares and pentagons.

(b) Show that $caba$ is not trivial by putting it into normal form. Show that $ab^{-1}ca^{-1}b$ is trivial, and that putting it into normal form corresponds to a null-homotopy crossing 4 cells; however, also show that there is a more efficient filling using only two cells.

(c) Prove a cubic upper bound on the Dehn function of $H(\mathbb{Z})$.

Exercise 4.2. We'll see that neither the visual nor horofunction boundary is nice for the Cayley graphs of free abelian groups, considering the case of $\mathbb{Z}^2$. Note that even though this is a CAT(0) group, its Cayley graph is not itself a CAT(0) space.

(a) Show that $\partial_\infty \mathsf{Cay}(\mathbb{Z}^2, \mathsf{std})$ is uncountable with the trivial topology. For the topology, you will use the observation that if N is the north-pointing ray from the origin and E_n is the ray that starts n steps east, then turns north forever, then $E_n \sim N$ for all n but $E_n \to E$.

(b) On the other hand, show that $\partial_h \mathsf{Cay}(\mathbb{Z}^2, \mathsf{std})$ can be realized as a countable subset of the circle with exactly four accumulation points. From the argument, you should also see why $\partial_h \mathsf{Cay}(\mathbb{Z}^2, S_{\mathsf{hex}})$ has six accumulation points—this illustrates that the horofunction boundary is not QI invariant.

Exercise 4.3. For a space of your choice, find a horofunction that is not a Busemann function. Can you do it in a δ–hyperbolic space?

Exercise 4.4. It is a fact that all horofunctions in the Euclidean plane (or CAT(0) spaces more generally) are Busemann functions—prove it if you like!

Show from the definition that the horocycles in $\mathbb{E}^2$ are lines perpendicular to the ray inducing the horofunction.

Exercise 4.5. Show that *for $F = F_{\{x_n\}} \in \partial_h$, the level set of F containing z is just the limit of larger and larger spheres through z centered at the points x_n.*

That is: if F is the horofunction $F_{\{x_n\}}$ defined by $F(y) = \lim_{n\to\infty} d(x_n, y) - d(x_n, x_0)$, then for any z the limit of the spheres $S_{d(z,x_n)}(x_n) =: L_n$ is the level set $L = \{y : F(y) = F(z)\}$.

If this gives you bad flashbacks to analysis classes, then just convince yourself of this fact in some pictures for the Euclidean plane case.

Exercise 4.6. What are the horocycles in $\mathbb{R}^2$ with the L^3 metric? How about $\mathbb{R}^2$ with the sup metric?

Exercises for Day 5

Lecture topics. Quasi-axes, translation length, dynamics, asymptotic density, and random groups.

Exercise 5.1. (D. Calegari) Consider the Baumslag-Solitar group $BS(m,n) = \langle a, b : ba^m b^{-1} = a^n \rangle$. Show that it can't be isomorphic to a subgroup of any hyperbolic group, with the following steps: First rule out the case $m = n$. Then for $m \neq n$, show that there is an element of the BS group with asymptotic translation length zero. Finally, in a hyperbolic group, explain why only torsion elements can have asymptotic translation length zero.

Exercise 5.2. In a geodesic space X, with basepoint x_0, let $\mathcal{G}(x)$ the set of all geodesics $\overline{x_0 x}$. Define an *instability function* $\mathcal{I}(x) := 2 \cdot \frac{diam\,\mathcal{G}(x)}{d(x,x_0)}$.

Check that for $(\mathbb{Z}^2, \mathsf{std})$, we have $\mathcal{I}(a,0) = 0$ but $\mathcal{I}(a,a) = 1$. For an arbitrary genset S, which are the directions with instability 0 and instability 1?

A sequence of points x_n can be called (sublinearly) stable if $\mathcal{I}(x_n) \to 0$. Explain why every proper sequence of points is stable in a geodesic hyperbolic space. Find a stable sequence and an unstable sequence in $H(\mathbb{Z}) = \langle a, b, c : [a,b] = c, [a,c] = [b,c] = 1. \rangle$.

Exercise 5.3. For $\mathbb{Z}^2$ with standard generators, compute $\mathrm{Prob}\big(\mathcal{I}(x) \leq \epsilon\big)$ exactly, as a function of ϵ. Prove that for any dimension d and finite genset S,

$$\mathsf{Stab}(\mathbb{Z}^d, S) = \lim_{\epsilon \to 0} \mathrm{Prob}\big(\mathcal{I}(x) \leq \epsilon\big) = 0.$$

Exercise 5.4. Suppose a chess-knight moves on an infinite board, and consider the ball of radius N: all positions reachable in N or fewer steps. Let A_N be the subset of those positions lying within 5° of the north axis from the starting point, and let D_N be the positions within 5° of the north-east axis from the starting point. The ratio $|A_N|/|D_N|$ converges to a rational number a/d. Compute it.

Exercise 5.5. Prove the "probabilistic pigeon-hole principle": if N^α balls are randomly put into N boxes, for any fixed $\alpha > 1/2$, then there is almost sure to be a box with at least two balls, as $N \to \infty$. (You'll set this up like the famous Birthday Problem, then use the Stirling approximation and a Taylor approximation or two.)

Exercise 5.6. Let R be a random set of relators at density d, at length ℓ. Fix $0 \leq \alpha < d$. Show that as $\ell \to \infty$, the probability that every reduced word of length $\alpha\ell$ appears as a subword of some word in R goes to 1.